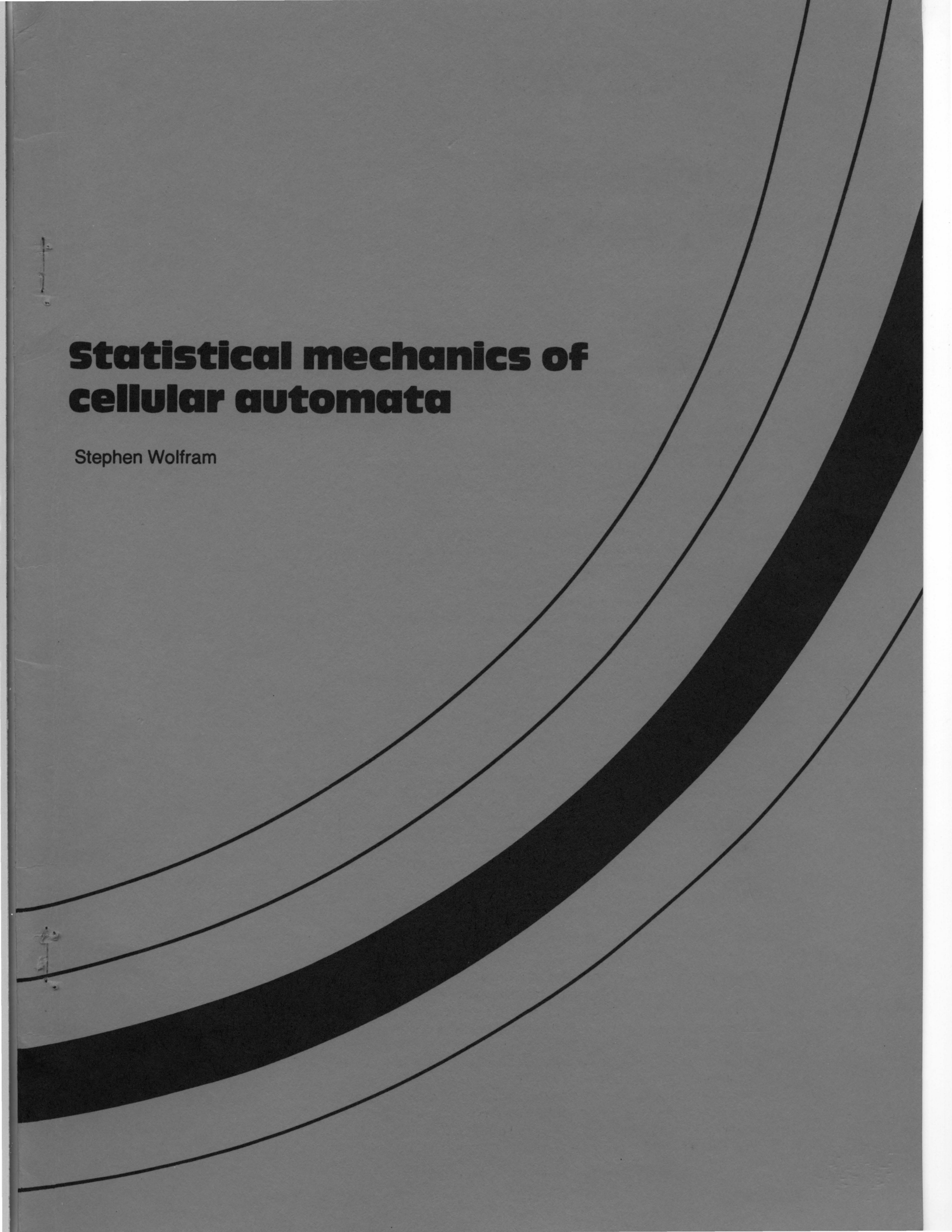# Statistical mechanics of cellular automata

Stephen Wolfram

Statistical mechanics of cellular automata

Stephen Wolfram

The Institute for Advanced Study, Princeton, New Jersey 08540

Cellular automata are used as simple mathematical models to investigate self-organization in statistical mechanics. A detailed analysis is given of "elementary" cellular automata consisting of a sequence of sites with values 0 or 1 on a line, with each site evolving deterministically in discrete time steps according to definite rules involving the values of its nearest neighbors. With simple initial configurations, the cellular automata either tend to homogeneous states, or generate self-similar patterns with fractal dimensions ≃1.59 or ≃1.69. With "random" initial configurations, the irreversible character of the cellular automaton evolution leads to several self-organization phenomena. Statistical properties of the structures generated are found to lie in two universality classes, independent of the details of the initial state or the cellular automaton rules. More complicated cellular automata are briefly considered, and connections with dynamical systems theory and the formal theory of computation are discussed.

CONTENTS

I. INTRODUCTION

The second law of thermodynamics implies that isolated microscopically reversible physical systems tend with time to states of maximal entropy and maximal "disorder." However, "dissipative" systems involving microscopic irreversibility, or those open to interactions with their environment, may evolve from "disordered" to more "ordered" states. The states attained often exhibit a complicated structure. Examples are outlines of snowflakes, patterns of flow in turbulent fluids, and biological systems. The purpose of this paper is to begin the investigation of cellular automata (introduced in Sec. II) as a class of mathematical models for such behavior. Cellular automata are sufficiently simple to allow detailed mathematical analysis, yet sufficiently complex to exhibit a wide variety of complicated phenomena. Cellular automata are also of sufficient generality to provide simple models for a very wide variety of physical, chemical, biological, and other systems. The ultimate goal is to abstract from a study of cellular automata general features of "self-organizing" behavior and perhaps to devise universal laws analogous to the laws of thermodynamics. This paper concentrates on the mathematical features of the simplest cellular automata, leaving for future study more complicated cellular automata and details of applications to specific systems. The paper is largely intended as an original contribution, rather than a review. It is presented in this journal in the hope that it may thereby reach a wider audience than would otherwise be possible. An outline of some of its results is given in Wolfram (1982a).

Investigations of simple "self-organization" phenomena in physical and chemical systems (Turing, 1952; Haken, 1975, 1978, 1979, 1981; Nicolis and Prigogine, 1977; Lan-

dauer, 1979; Prigogine, 1980; Nicolis *et al.*, 1981) have often been based on the Boltzmann transport differential equations (e.g., Lifshitz and Pitaevskii, 1981) (or its analogs) for the time development of macroscopic quantities. The equations are obtained by averaging over an ensemble of microscopic states and assuming that successive collisions between molecules are statistically uncorrelated. For closed systems (with reversible or at least unitary microscopic interactions) the equations lead to Boltzmann's H theorem, which implies monotonic evolution towards the macroscopic state of maximum entropy. The equations also imply that weakly dissipative systems (such as fluids with small temperature gradients imposed) should tend to the unique condition of minimum entropy production. However, in strongly dissipative systems, several final states may be possible, corresponding to the various solutions of the polynomial equations obtained from the large time limit of the Boltzmann equations. Details or "fluctuations" in the initial state determine which of several possible final states are attained, just as in a system with multiple coexisting phases. Continuous changes in parameters such as external concentrations or temperature gradients may lead to discontinuous changes in the final states when the number of real roots in the polynomial equations changes, as described by catastrophe theory (Thom, 1975). In this way, "structures" with discrete boundaries may be formed from continuous models. However, such approaches become impractical for systems with very many degrees of freedom, and therefore cannot address the formation of genuinely complex structures.

More general investigations of self-organization and "chaos" in dynamical systems have typically used simple mathematical models. One approach (e.g., Ott, 1981) considers dissipative nonlinear differential equations (typically derived as idealizations of Navier-Stokes hydrodynamic equations). The time evolution given particular initial conditions is represented by a trajectory in the space of variables described by the differential equations. In the simplest cases (such as those typical for chemical concentrations described by the Boltzmann transport equations), all trajectories tend at large times to a small number of isolated limit points, or approach simple periodic limit cycle orbits. In other cases, the trajectories

may instead concentrate on complicated and apparently chaotic surfaces ("strange attractors"). Nearly linear systems typically exhibit simple limit points or cycles. When nonlinearity is increased by variation of external parameters, the number of limit points or cycles may increase without bound, eventually building up a strange attractor (typically exhibiting a statistically self-similar structure in phase space). A simpler approach (e.g., Ott, 1981) involves discrete time steps, and considers the evolution of numbers on an interval of the real line under iterated mappings. As the nonlinearity is increased, greater numbers of limit points and cycles appear, followed by essentially chaotic behavior. Quantitative features of this approach to chaos are found to be universal to wide classes of mappings. Notice that for both differential equations and iterated mappings, initial conditions are specified by real numbers with a potentially infinite number of significant digits. Complicated or seemingly chaotic behavior is a reflection of sensitive dependence on high-order digits in the decimal expansions of the numbers.

Models based on cellular automata provide an alternative approach, involving discrete coordinates and variables as well as discrete time steps. They exhibit complicated behavior analogous to that found with differential equations or iterated mappings, but by virtue of their simpler construction are potentially amenable to a more detailed and complete analysis.

Section II of this paper defines and introduces cellular automata and describes the qualitative behavior of elementary cellular automata. Several phenomena characteristic of self-organization are found. Section III gives a quantitative statistical analysis of the states generated in the time evolution of cellular automata, revealing several quantitative universal features. Section IV describes the global analysis of cellular automata and discusses the results in the context of dynamical systems theory and the formal theory of computation. Section V considers briefly extensions to more complicated cellular automata. Finally, Sec. VI gives some tentative conclusions.

II. INTRODUCTION TO CELLULAR AUTOMATA

Cellular automata are mathematical idealizations of physical systems in which space and time are discrete, and physical quantities take on a finite set of discrete values. A cellular automaton consists of a regular uniform lattice (or "array"), usually infinite in extent, with a discrete variable at each site ("cell"). The state of a cellular automaton is completely specified by the values of the variables at each site. A cellular automaton evolves in discrete time steps, with the value of the variable at one site being affected by the values of variables at sites in its "neighborhood" on the previous time step. The neighborhood of a site is typically taken to be the site itself and all immediately adjacent sites. The variables at each site are updated simultaneously ("synchronously"), based on the values of the variables in their neighborhood at the preceding time step, and according to a definite set of "local rules."

Cellular automata were originally introduced by von Neumann and Ulam (under the name of "cellular spaces") as a possible idealization of biological systems (von Neumann, 1963, 1966), with the particular purpose of modelling biological self-reproduction. They have been applied and reintroduced for a wide variety of purposes, and referred to by a variety of names, including "tessellation automata," "homogeneous structures," "cellular structures," "tessellation structures," and "iterative arrays."

Physical systems containing many discrete elements with local interactions are often conveniently modelled as cellular automata. Any physical system satisfying differential equations may be approximated as a cellular automaton by introducing finite differences and discrete variables.[1] Nontrivial cellular automata are obtained whenever the dependence on the values at each site is nonlinear, as when the system exhibits some form of "growth inhibition." A very wide variety of examples may be considered; only a few are sketched here. In the most direct cases, the cellular automaton lattice is in position space. At a microscopic level, the sites may represent points in a crystal lattice, with values given by some quantized observable (such as spin component) or corresponding to the types of atoms or units. The dynamical Ising model (with kinetic energy terms included) and other lattice spin systems are simple cellular automata, made nondeterministic by "noise" in the local rules at finite temperature. At a more macroscopic level, each site in a cellular automaton may represent a region containing many molecules (with a scale size perhaps given by an appropriate correlation length), and its value may label one of several discrete possible phases or compositions. In this way, cellular automata may be used as discrete models for nonlinear chemical systems involving a network of reactions coupled with spatial diffusion (Greenberg et al., 1978). They have also been used in a (controversial) model for the evolution of spiral galaxies (Gerola and Seiden, 1978; Schewe, 1981). Similarly, they may provide models for kinetic aspects of phase transitions (e.g., Harvey et al., 1982). For example, it is possible that growth of dendritic crystals (Langer, 1980) may be described by aggregation of discrete "packets" with a local growth inhibition effect associated with local releases of latent heat, and thereby treated as a cellular automaton [Witten and Sander (1981) discuss a probabilistic model of this kind, but there are indications that the probabilistic elements are inessential]. The spatial structure of turbulent fluids may perhaps be modelled using cellular automata by approximating the velocity field as a lattice of cells, each containing one or no eddies, with interactions between neighboring cells. Physical systems may also potentially be described by cellular automata in wave-vector or momentum space, with site values representing excitations in the corresponding modes.

---

[1] The discussion here concentrates on systems first order in time; a more general case is mentioned briefly in Sec. IV.

Many biological systems have been modelled by cellular automata (Lindenmayer, 1968; Herman, 1969; Ulam, 1974; Kitagawa, 1974; Baer and Martinez, 1974; Rosen, 1981) (cf. Barricelli, 1972). The development of structure and patterns in the growth of organisms often appears to be governed by very simple local rules (Thompson, 1961; Stevens, 1974) and is therefore potentially well described by a cellular automaton model. The discrete values at each site typically label types of living cells, approximated as growing on a regular spatial lattice. Short-range or contact interactions may lead to expression of different genetic characteristics, and determine the cell type. Simple nonlinear rules may lead to the formation of complex patterns, as evident in many plants and animals. Examples include leaf and branch arrangements (e.g., Stevens, 1974) and forms of radiolarian skeletons (e.g., Thompson, 1961). Simple behavior and functioning of organisms may be modelled by cellular automata with site values representing states of living cells or groups of cells [Burks (1973) and Flanigan (1965) discuss an example in heart fibrillation]. The precise mathematical formulation of such models allows the behavior possible in organisms or systems with particular construction or complexity to be investigated and characterized (e.g., von Neumann, 1966). Cellular automata may also describe populations of non-mobile organisms (such as plants), with site values corresponding to the presence or absence of individuals (perhaps of various types) at each lattice point, with local ecological interactions.

Cellular automata have also been used to study problems in number theory and their applications to tapestry design (Miller, 1970, 1980; ApSimon, 1970a, 1970b; Sutton, 1981). In a typical case, successive differences in a sequence of numbers (such as primes) reduced with a small modulus are taken, and the geometry of zero regions is investigated.

As will be discussed in Sec. IV, cellular automata may be considered as parallel processing computers (cf. Manning, 1977; Preston *et al.*, 1979). As such, they have been used, for example, as highly parallel multipliers (Atrubin, 1965; Cole, 1969), sorters (Nishio, 1981), and prime number sieves (Fischer, 1965). Particularly in two dimensions, cellular automata have been used extensively for image processing and visual pattern recognition (Deutsch, 1972; Sternberg, 1980; Rosenfeld, 1979). The computational capabilities of cellular automata have been studied extensively (Codd, 1968; Burks, 1970; Banks, 1971; Aladyev, 1974, 1976; Kosaraju, 1974; Toffoli, 1977b), and it has been shown that some cellular automata could be used as general purpose computers, and may therefore be used as general paradigms for parallel computation. Their locality and simplicity might ultimately permit their implementation at a molecular level.

The notorious solitaire computer game "Life" (Conway, 1970; Gardner, 1971, 1972; Wainwright, 1971–1973; Wainwright, 1974; Buckingham, 1978; Berlekamp *et al.*, 1982; R. W. Gosper, private communications) (qualitatively similar in some respects to the game of "Go") is an example of a two-dimensional cellular automaton, to be discussed briefly in Sec. V.

Until Sec. V, we shall consider exclusively one-dimensional cellular automata with two possible values of the variables at each site ("base 2") and in which the neighborhood of a given site is simply the site itself and the sites immediately adjacent to it on the left and right. We shall call such cellular automata elementary. Figure 1 specifies one particular set of local rules for an elementary cellular automaton. On the top row, all $2^3 = 8$ possible values of the three variables in the neighborhood are given, and below each one is given the value achieved by the central site on the next time step according to a particular local rule. Figure 2 shows the evolution of a particular state of the cellular automaton through one time step according to the local rule given in Fig. 1.

The local rules for a one-dimensional neighborhood-three cellular automaton are described by an eight-digit binary number, as in the example of Fig. 1. (In specifying cellular automata, we use this binary number interchangeably with its decimal equivalent.) Since any eight-digit binary number specifies a cellular automaton, there are $2^8 = 256$ possible distinct cellular automaton rules in one dimension with a three-site neighborhood. Two inessential restrictions will usually be imposed on these rules. First, a cellular automaton rule will be considered "illegal" unless a "null" or "quiescent" initial state consisting solely of 0 remains unchanged. This forbids rules whose binary specification ends with a 1 (and removes symmetry in the treatment of 0 and 1 sites). Second, the rules must be reflection symmetric, so that 100 and 001 (and 110 and 011) yield identical values. These restrictions[2] leave 32 possible "legal" cellular automaton rules of the form $\alpha_1 \alpha_2 \alpha_3 \alpha_4 \alpha_2 \alpha_5 \alpha_4 0$.

The local rules for a cellular automaton may be considered as a Boolean function of the sites within the neighborhood. Let $s_n(m)$ be the value of site m at time step n. As a first example consider the "modulo-two" rule 90 (also used as the example for Fig. 1). According to this rule, the value of a particular site is simply the sum modulo two of the values of its two neighboring sites on the previous time step. The Boolean equivalent of this rule is therefore

$$s_{n+1}(m) = s_n(m-1) \oplus s_n(m+1) \qquad (2.1)$$

or schematically $s_+ = s^- \oplus s^+$, where $\oplus$ denotes addition modulo two ("exclusive disjunction" or "inequality"). Similarly, rule 18 is equivalent to $s_+ = s \vee (s^- \oplus s^+)$ [where s denotes $s_n(m)$], rule 22 to $s_+ = s \vee (s^- \wedge s^+)$, rule 54 to $s_+ = s \oplus (s^- \vee s^+)$, rule 150 to $s_+ = s^- \oplus s \oplus s^+$, and so on. Designations s^- and s^+ always enter symmetrically in legal cellular automaton rules by virtue of re-

---

[2] The quiescence condition is required in many applications to forbid "instantaneous propagation" of value-one sites. The reflection symmetry condition guarantees isotropy as well as homogeneity in cellular automaton evolution.

111	110	101	100	011	010	001	000
0	1	1	0	1	0	1	0

FIG. 1. Example of a set of local rules for the time evolution of a one-dimensional elementary cellular automaton. The variables at each site may take values 0 or 1. The eight possible states of three adjacent sites are given on the upper line. The lower line then specifies a rule for the time evolution of the cellular automaton by giving the value to be taken by the central site of the three on the next time step. The time evolution of the complete cellular automaton is obtained by simultaneous application of these rules at each site for each time step. The rule given is the modulo-two rule: the value of a site at a particular time step is simply the sum modulo two of the values of its two neighbors at the previous time step. Any possible sequence of eight binary digits specifies a cellular automaton.

flection symmetry. The Boolean function representation of cellular automaton rules is convenient for practical implementation on standard serial processing digital computers.[3]

Some cellular automaton rules exhibit the important simplifying feature of "additive superposition" or "additivity." Evolution according to such rules satisfies the superposition principle

$$s_0 = t_0 \oplus u_0 \leftrightarrow s_n = t_n \oplus u_n, \qquad (2.2)$$

which implies that the configurations obtained by evolution from any initial configuration are given by appropriate combinations of those found in Fig. 3 for evolution from a single nonzero site. Notice that such additivity does not imply linearity in the real number sense of Sec. I, since the addition is over a finite field. Cellular automata satisfy additive superposition only if their rule is of the form $\alpha_1 \alpha_2 0 \alpha_3 \alpha_2 \alpha_1 \alpha_3 0$ with $\alpha_3 = \alpha_1 \oplus \alpha_2$. Only rules 0, 90, 150, and 204 are of this form. Rules 0 and 204 are trivial; 0 erases any initial configuration, and 204 maintains any initial configuration unchanged (performing the identity transformation at each time step). Rule 90 is the modulo-two rule discussed above, and takes a particular site to be the sum modulo two of the values of its two neighbors at the previous time step, as in Eq. (2.1). Rule 150 is similar. It takes a particular site to be the sum modulo two of the values of its two neighbors and its own

[3] The values of a sequence of (typically 32) sites are represented by bits in a single computer word. Copies of this word shifted one bit to the left and one bit to the right are obtained. Then the cellular automaton rule may be applied in parallel to all bits in the words using single machine instructions for each wordwise Boolean operation. An analogous procedure is convenient in simulation of two-dimensional cellular automata on computer systems with memory-mapped displays, for which application of identical Boolean operations to each display pixel is usually implemented in hardware or firmware.

0	1	0	1	0	1	0	1	0	1	0	1	1	0	0	0
	0	0	1	1	0	1	1	0	0	0	0	1	0	1	0	1	0

FIG. 2. Evolution of a configuration in one-dimensional cellular automaton for one time step according to the modulo-two rule given in Fig. 1. The values of the two end sites after the time step depend on the values of sites not shown here.

value at the previous time step ($s_+ = s^- \oplus s \oplus s^+$).

The additive superposition principle of Eq. (2.2) combines values at different sites by addition modulo two (exclusive disjunction). Combining values instead by conjunction (Boolean multiplication) yields a superposition principle for rules 0, 4, 50, and 254. Combining values by (inclusive) disjunction (Boolean addition) yields a corresponding principle for rules 0, 204, 250, and 254. It is found that no other legal cellular automaton rules satisfy superposition principles with any combining function.

The Boolean representation of cellular automaton rules reveals that some rules are "peripheral" in the sense that the value of a particular site depends on the values of its two neighbors at the previous time step, but not on its own previous value. Rules 0, 90, 160, and 250 are of the form $\alpha_1 \alpha_2 \alpha_1 \alpha_2 \alpha_2 0 \alpha_2 0$ and exhibit this property.

Having discussed features of possible local rules we now outline their consequences for the evolution of elementary cellular automata. Sections III and IV present more detailed quantitative analysis.

Figure 3 shows the evolution of all 32 possible legal cellular automata from an initial configuration containing a single site with value 1 (analogous to the growth of a "crystal" from a microscopic "seed"). The evolution is shown until a particular configuration appears for the second time (a "cycle" is detected), or for at most 20 time steps. Several classes of behavior are evident. In one class, the initial 1 is immediately erased (as in rules 0 and 160), or is maintained unchanged forever (as in rules 4 and 36). Rules of this class are distinguished by the presence of the local rules $100 \rightarrow 0$ and $001 \rightarrow 0$, which prevent any propagation of the initial 1. A second class of rules (exemplified by 50 or 122) copies the 1 to generate a uniform structure which expands by one site in each direction on each time step. These two classes of rules will be termed "simple." A third class of rules, termed "complex," and exemplified by rules 18, 22, and 90, yields nontrivial patterns.

As a consequence of their locality, cellular automaton rules define no intrinsic length scale other than the size of a single site (or of a neighborhood of three sites) and no intrinsic time scale other than the duration of a single time step. The initial state consisting of a single site with value 1 used in Fig. 3 also exhibits no intrinsic scale. The cellular automaton configurations obtained in Fig. 3 should therefore also exhibit no intrinsic scale, at least in the infinite time limit. Simple rules yield a uniform final state, which is manifestly scale invariant. The scale invariance of the configurations generated by complex rules
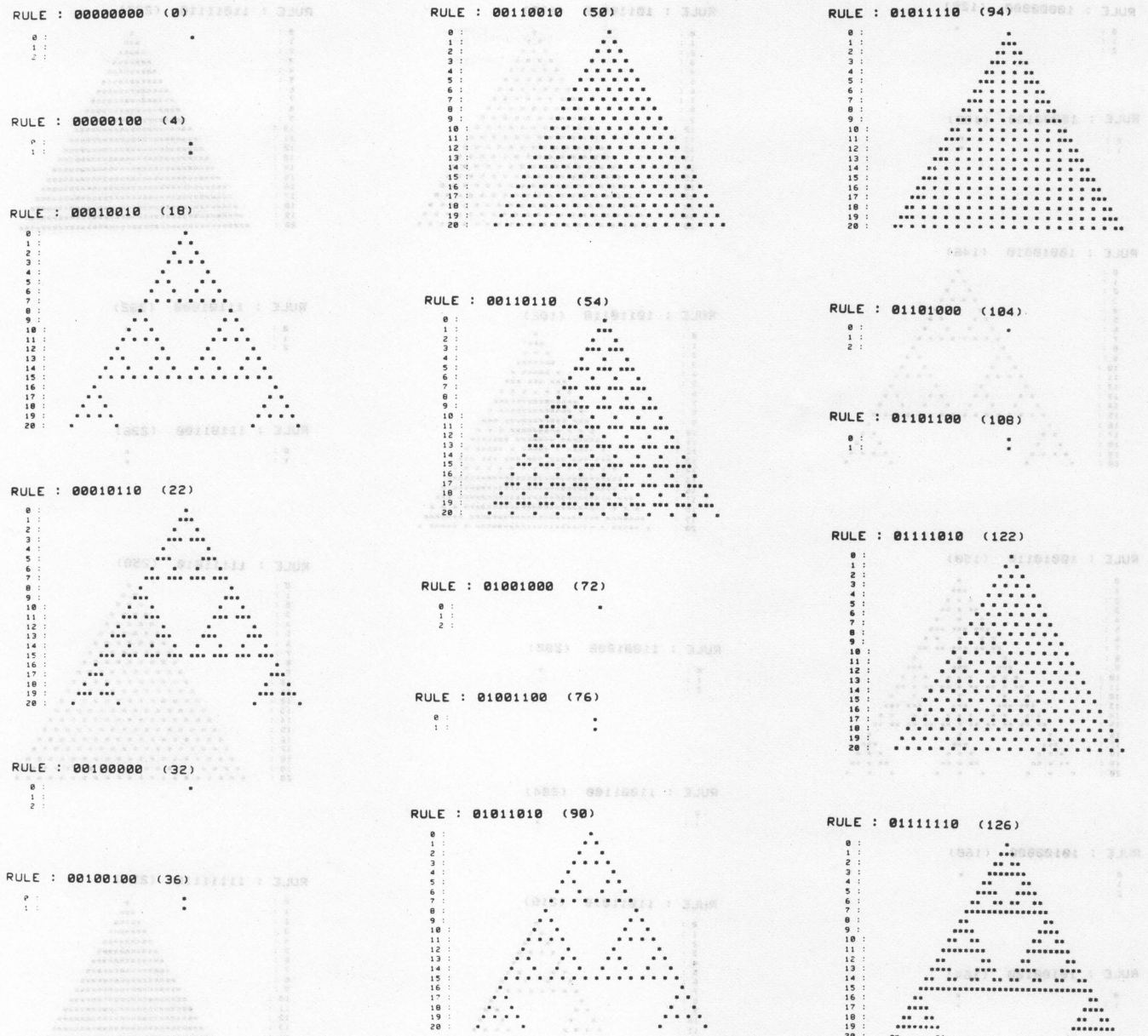
FIG. 3. Evolution of one-dimensional elementary cellular automata according to the 32 possible legal sets of rules, starting from a state containing a single site with value 1. Sites with value 1 are represented by stars, and those with value 0 by blanks. The configurations of the cellular automata at successive time steps are shown on successive lines. The time evolution is shown up to the point where the system is detected to cycle (visiting a particular configuration for the second time), or for at most 20 time steps. The process is analogous to the growth of a crystal from a microscope seed. A considerable variety of behavior is evident. The cellular automata which do not tend to a uniform state yield asymptotically self-similar fractal configurations.

is nontrivial. In the infinite time limit, the configurations are "self-similar" in that views of the configuration with different "magnifications" (but with the same "resolution") are indistinguishable. The configurations thus exhibit the same structure on all scales.

Consider as an example the modulo-two rule 90 (also used as the example for Fig. 1 and in the discussion above). This rule takes each site to be the sum modulo two of its two nearest neighbors on the previous time step. Starting from an initial state containing a single site with value 1, the configuration it yields on successive time steps is thus simply the lines of Pascal's triangle modulo

two, as illustrated in Fig. 4 (cf. Wolfram, 1982b). The values of the sites are hence the values of binomial coefficients [or equivalently, coefficients of x^i in the expansion of $(1+x)^n$] modulo two. In the large time limit, the pattern of sites with value 1 may be obtained by the recursive geometrical construction (cf. Sierpinski, 1916; Abelson and diSessa, 1981, Sec. 2.4) shown in Fig. 5. This geometrical construction manifests the self-similarity (Mandelbrot, 1977, 1982; Geffen *et al.*, 1981) or "scale invariance" of the resulting curve. Figure 3 shows that evolution of other complex cellular automata from a single nonzero site yields essentially identical self-similar pat-
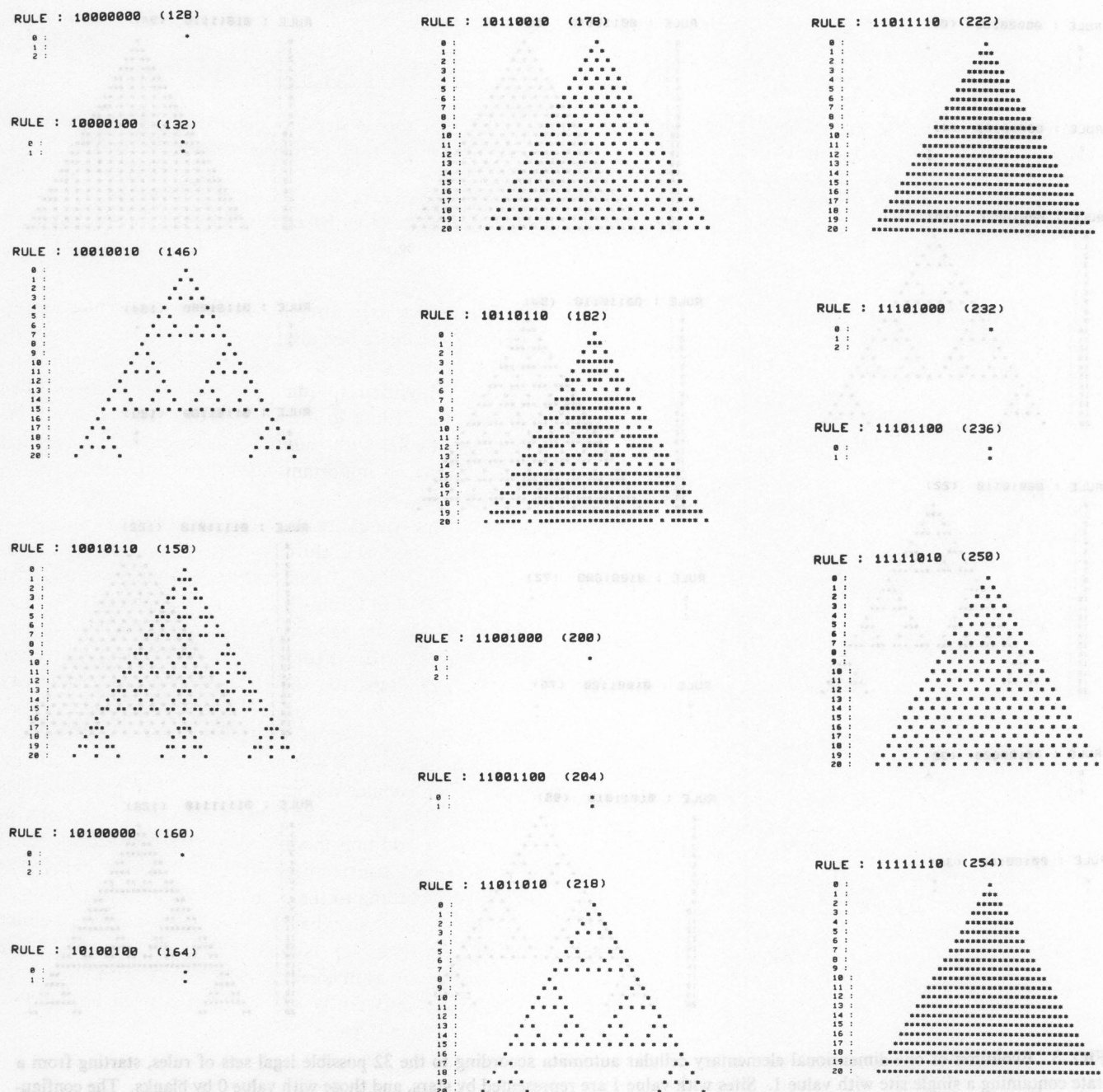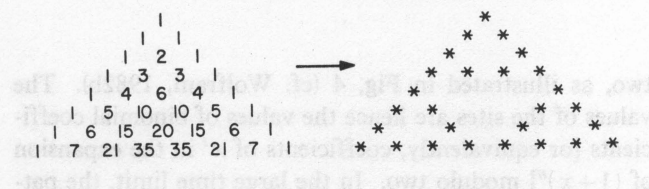
FIG. 3. (*Continued.*)

FIG. 4. An algebraic construction for the configurations of a cellular automaton starting from a state containing a single site with value 1 and evolving according to the modulo-two rule 90. The rule is illustrated in Fig. 1, and takes the value of a particular site to be the sum modulo two of the values of its two neighboring sites at the previous time step. The value of a site at a given time step is then just the value modulo two of the corresponding binomial coefficient in Pascal's triangle.

terns. An exception is rule 150, for which the value of each site is determined by the sum modulo two of its own value and the values of its two neighbors on the previous time step. The sequence of binary digits obtained by evolution from a single-site initial state for n time steps with this rule is thus simply the coefficients of x^i in the expansion of $(x^2 + x + 1)^n$ modulo two. A geometrical construction for the pattern obtained is given in Fig. 6.

Figure 7 shows examples of time evolution for some cellular automata with illegal local rules (defined above) which were omitted from Fig. 3. When the quiescence condition is violated, successive time steps involve alternation of 0 and 1 at infinity. When reflection symmetry is violated, the configurations tend to undergo uniform

FIG. 5. Sequence of steps in a geometrical construction for the large time behavior of a cellular automaton evolving according to the modulo-two rule 90. The final pattern is the limit of the sequence shown here. It is a self-similar figure with fractal dimension $\log_2 3$.

FIG. 6. Sequence of steps in a geometrical construction for the large time behavior of a cellular automaton evolving according to the modulo-two rule 150. The final pattern is the limit of the sequence shown here. It is a self-similar figure with fractal dimension $\log_2 2\varphi \simeq 1.69$ [where $\varphi = (1+\sqrt{5})/2$ is the golden ratio]. An analogous construction for rule 90 was given in Fig. 5.

shifting. The self-similar patterns seen in Fig. 3 are also found in cases such as rule 225, but are sheared by the overall shifting. It appears that consideration of illegal as well as "legal" cellular automaton rules introduces no qualitatively new features.

Figure 3 showed the growth of patterns by cellular automaton evolution from a very simple initial state containing a single nonzero site (seed). Figure 8 now illustrates time evolution from a disordered or "random" initial state according to each of the 32 legal cellular automaton rules. A specific "typical" initial configuration was taken, with the value of each site chosen independently, with equal probabilities for values 0 and 1.[4] Just as in Fig. 3, several classes of behavior are evident. The simple rules exhibit trivial behavior, either yielding a uniform final state or essentially preserving the form of the initial state. Complex rules once again yield nontrivial behavior. Figure 8 illustrates the remarkable fact that time evolution according to these rules destroys the independence of the initial sites, and generates correlations between values at separated sites. This phenomenon is the essence of self-organization in cellular automata. An initially random state evolves to a state containing long-range correlations and structure. The bases of the "triangles" visible in Fig. 8 are fluctuations in which a sequence of many adjacent cells have the same value. The length of these correlated sequences is reduced by one site per time step, yielding the distinctive triangular structure. Figure 8 suggests that triangles of all sizes are generated. Section III confirms this impression through a quantitative analysis and discusses universal features of the structures obtained.

The behavior of the cellular automata shown in Fig. 8 may be characterized in analogy with the behavior of dynamical systems (e.g., Ott, 1981): simple rules exhibit simple limit points or limit cycles, while complex rules exhibit phenomena analogous to strange attractors.

The cellular automata shown in Fig. 8 were all assumed to satisfy periodic boundary conditions. Instead of treating a genuinely infinite line of sites, the first and last sites

are identified, as if they lay on a circle of finite radius. Cellular automata can also be rendered finite by imposing null boundary conditions, under which sites beyond each end are modified to maintain value zero, rather than evolving according to the local rules. Figure 9 compares results obtained with these two boundary conditions in a simple case; no important qualitative differences are apparent.

Finite one-dimensional cellular automata are similar to a class of feedback shift registers (e.g., Golomb, 1967; Berlekamp, 1968).[5] A feedback shift register consists of a sequence of sites ("tubes") carrying values $a(i)$. At each time step, the site values evolve by a shift $a(i) = a(i-1)$ and feedback $a(0) = \mathbf{F}[a(j_1), a(j_2), \ldots]$ where, j_i give the positions of "taps" on the shift register. An elementary cellular automaton of length N corresponds to a feedback shift register of length N with site values 0 and 1 and taps at positions $N-2$, $N-1$, and N. The Boolean function F defines the cellular automaton rule. [The additive rules 90 and 150 correspond to linear feedback shift registers in which $\mathbf{F}$ is addition modulo two (exclusive disjunction).] At each shift register time step, the value of one site is updated according to the cellular automaton rule. After N time steps, all N sites have been updated, and one cellular automaton time step is complete. All interior sites are treated exactly as in a cellular automaton, but the two end sites evolve differently (their values depend on the two preceding time steps).

III. LOCAL PROPERTIES OF ELEMENTARY CELLULAR AUTOMATA

We shall examine now the statistical analysis of configurations generated by time evolution of "elementary" cellular automata, as illustrated in Figs. 3 and 8. This section considers statistical properties of individual such configurations; Sec. IV discusses the ensemble of all possible configurations. The primary purpose is to obtain a quantitative characterization of the "self-organization" pictorially evident in Fig. 8.

---

[4] Here and elsewhere a standard linear congruential pseudorandom number generator with recurrence relation $x_{n+1} = (1103515245 x_n + 12345) \bmod 2^{31}$ was used. Results were also obtained using other pseudorandom number generation procedures and using random numbers derived from real-time properties of a time-shared computer system.

[5] This similarity may be used as the basis for a simple hardware implementation of one-dimensional cellular automata (Pearson et al., 1981; Hoogland et al., 1982; Toffoli, 1983).

RULE : 00000001 (1)

RULE : 00000010 (2)

RULE : 00001011 (11)

RULE : 00001101 (13)

RULE : 00011100 (28)

RULE : 00011110 (30)

RULE : 11100001 (225)

RULE : 11111111 (255)

FIG. 7. Evolution of a selection of one-dimensional elementary cellular automata obeying illegal rules. Rules are considered illegal if they violate reflection symmetry, which requires identical rules for 100 and 001 and for 110 and 011, or if they violate the quiescence condition which requires that an initial state containing only 0 sites should remain unchanged. For example, rule 2 violates reflection symmetry, and thus yields a uniformly shifting pattern, while rule 1 violates the quiescence condition and yields a pattern which "flashes" from all 0 to all 1 in successive time steps.

A configuration may be considered disordered (or essentially random) if values at different sites are statistically uncorrelated (and thus behave as "independent random variables"). Such configurations represent a discrete form of "white noise." Deviations of statistical measures for cellular automaton configurations from their values for corresponding disordered configurations indicate order, and signal the presence of correlations between values at different sites. An (infinite) disordered configuration is specified by a single parameter, the independent probability p for each site to have value 1. The description of an ordered configuration requires more parameters.

Figure 10 shows a set of examples of disordered configurations with probabilities $p = 0.25$, 0.5, and 0.75. Such disordered configurations were used as the initial configurations for the cellular automaton evolution shown in Fig.
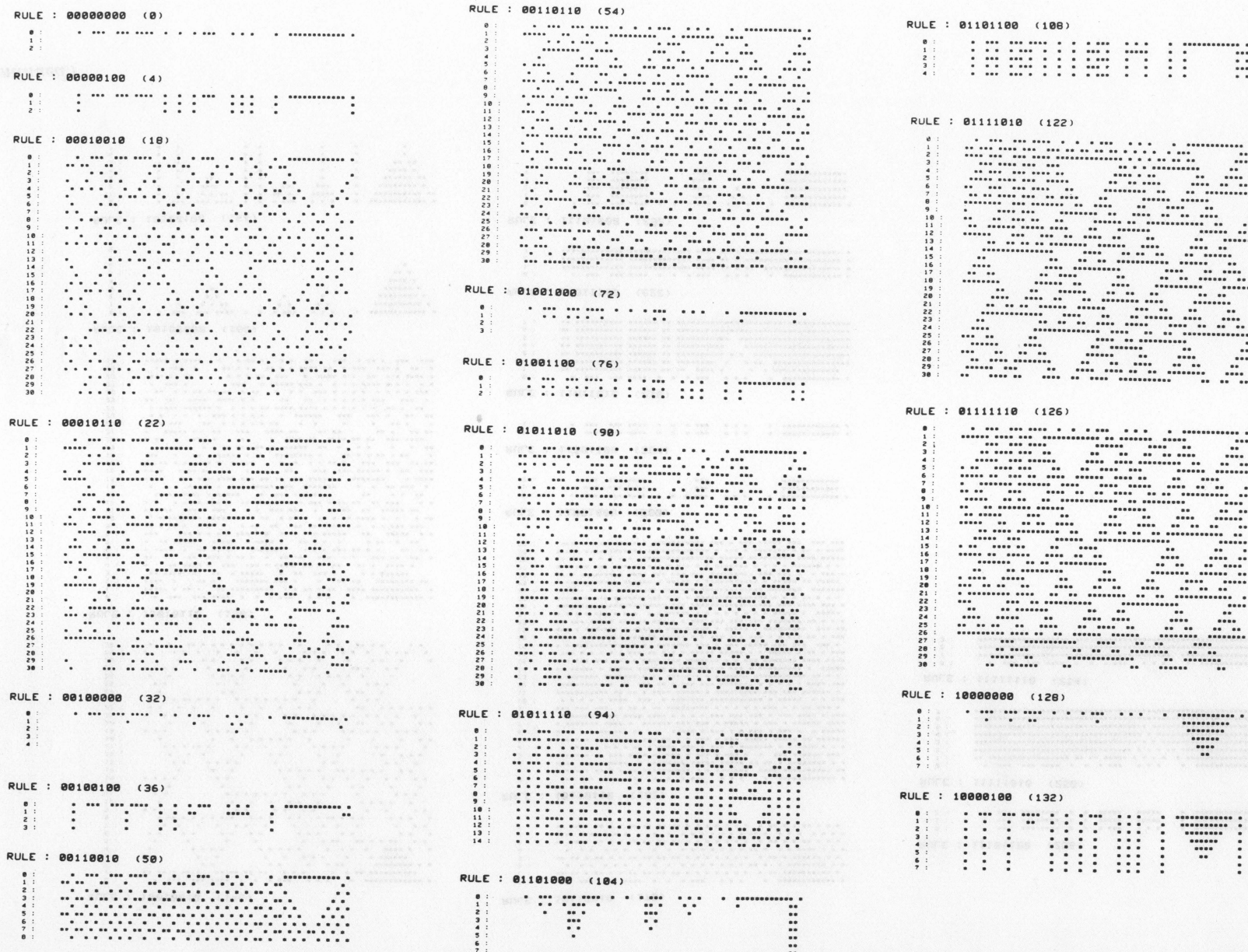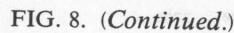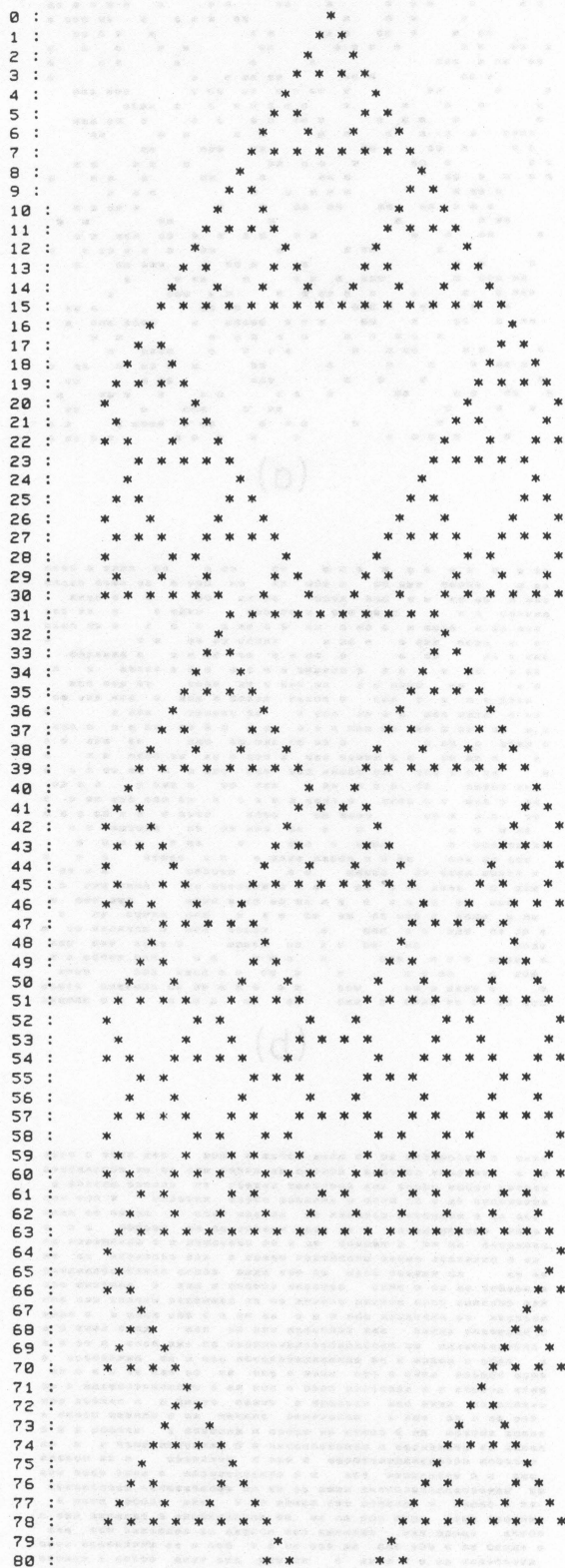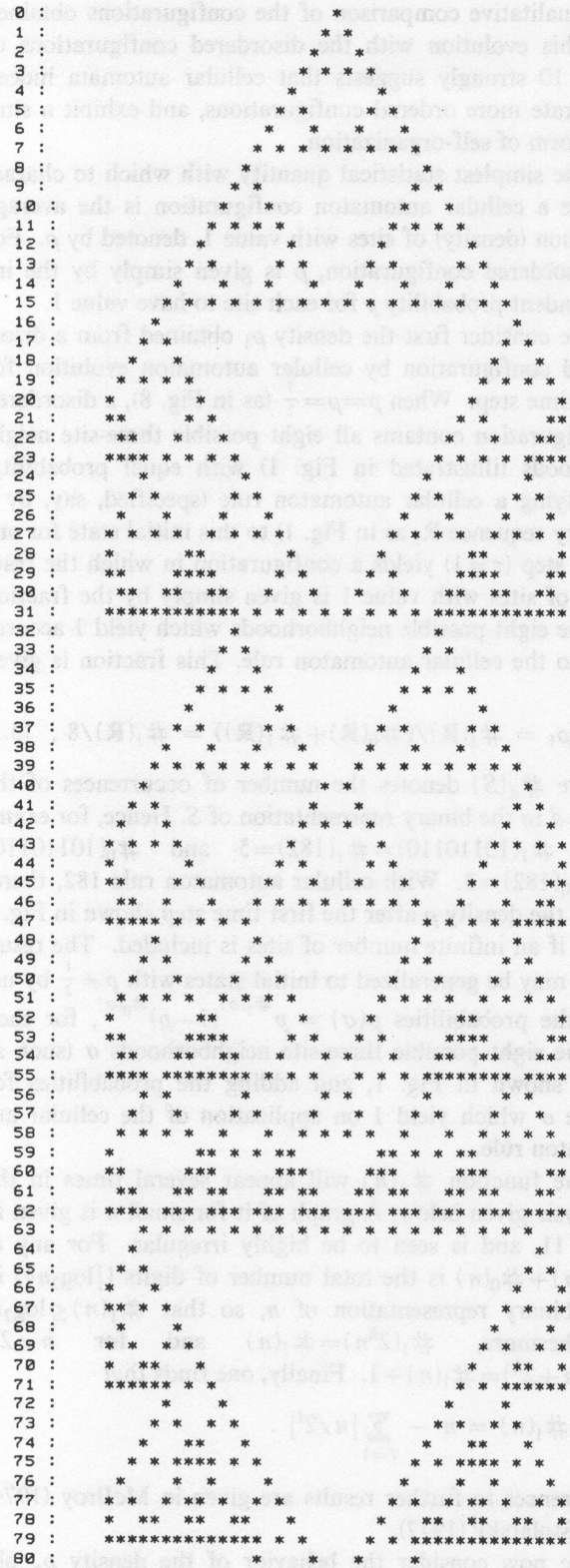
FIG. 8. Evolution of a disordered (random) initial state in each of the 32 possible legal one-dimensional elementary cellular automata. The value of each site is initially uncorrelated, and is taken to be 0 or 1 with probability $\frac{1}{2}$. Evolution is shown until a particular configuration appears for the second time, or for at most 30 time steps. Just as in Fig. 3, several classes of behavior are evident. In one class, time evolution generates long-range correlations and fluctuations, yielding distinctive "triangular" structures, and exhibiting a simple form of self-organization. All the cellular automata shown are taken to satisfy periodic boundary conditions, so that their sites are effectively arranged on a circle.

RULE : 10010010   (146)

RULE : 10010110   (150)

RULE : 10100000   (160)

RULE : 10100100   (164)

RULE : 10110010   (178)

RULE : 10110110   (182)

RULE : 11001000   (200)

RULE : 11001100   (204)

RULE : 11011010   (218)

RULE : 11011110   (222)

RULE : 11101000   (232)

RULE : 11101100   (236)

RULE : 11111010   (250)

RULE : 11111110   (254)

FIG. 8. (Continued.)

(a)

(b)

FIG. 9. Time evolution of a simple initial state according to the modulo-two rule 90, on a line of sites satisfying (a) periodic boundary conditions (so that first and last sites are identified, and the sites are effectively arranged on a circle), and (b) null boundary conditions (so that sites not shown are assumed always to have value 0). Changes in boundary conditions apparently have no significant qualitative effect.

8. Qualitative comparison of the configurations obtained by this evolution with the disordered configurations of Fig. 10 strongly suggests that cellular automata indeed generate more ordered configurations, and exhibit a simple form of self-organization.

The simplest statistical quantity with which to characterize a cellular automaton configuration is the average fraction (density) of sites with value 1, denoted by ρ. For a disordered configuration, ρ is given simply by the independent probability p for each site to have value 1.

We consider first the density ρ_1 obtained from a disordered configuration by cellular automaton evolution for one time step. When $p = \rho = \frac{1}{2}$ (as in Fig. 8), a disordered configuration contains all eight possible three-site neighborhoods (illustrated in Fig. 1) with equal probability. Applying a cellular automaton rule (specified, say, by a binary sequence $\mathbf{R}$, as in Fig. 1) to this initial state for one time step ($\tau = 1$) yields a configuration in which the fraction of sites with value 1 is given simply by the fraction of the eight possible neighborhoods which yield 1 according to the cellular automaton rule. This fraction is given by

$$\rho_1 = \#_1(\mathbf{R})/(\#_0(\mathbf{R}) + \#_1(\mathbf{R})) = \#_1(\mathbf{R})/8 , \quad (3.1)$$

where $\#_d(S)$ denotes the number of occurrences of the digit d in the binary representation of S. Hence, for example, $\#_1(10110110) = \#_1(182) = 5$ and $\#_0(10110110) = \#_0(182) = 3$. With cellular automaton rule 182, therefore, the density ρ after the first time step shown in Fig. 8 is $\frac{5}{8}$ if an infinite number of sites is included. The result (3.1) may be generalized to initial states with $p \neq \frac{1}{2}$ by using the probabilities $p(\sigma) = p^{\#_1(\sigma)}(1-p)^{\#_0(\sigma)}$, for each of the eight possible three-site neighborhoods σ (such as 110) shown in Fig. 1, and adding the probabilities for those σ which yield 1 on application of the cellular automaton rule.

The function $\#_1(n)$ will appear several times in the analysis given below. A graph of it for small n is given in Fig. 11, and is seen to be highly irregular. For any n, $\#_1(n) + \#_0(n)$ is the total number of digits ($\lceil \log_2 n \rceil$) in the binary representation of n, so that $\#_1(n) \leq \log_2 n$. Furthermore, $\#_1(2^k n) = \#_1(n)$ and for $n < 2^k$, $\#_1(n + 2^k) = \#_1(n) + 1$. Finally, one finds that

$$\#_1(n) = n - \sum_{i=1} \lfloor n/2^i \rfloor .$$

References to further results are given in McIlroy (1974) and Stolarsky (1977).

We now consider the behavior of the density ρ_τ obtained after τ time steps in the limit of large τ. When $\tau > 1$, correlations induced by cellular automaton evolution invalidate the approach used in Eq. (3.1), although a similar approach may nevertheless be used in deriving statistical approximations, as discussed below.

Figure 8 suggests that with some simple rules (such as 0, 32, or 72), any initial configuration evolves ultimately to the null state $\rho = 0$, although the length of transient varies. For rule 0, it is clear that $\rho = 0$ for all $\tau > 0$. Similarly, for rule 72, $\rho = 0$ for $\tau > 1$. For rule 32, infinite
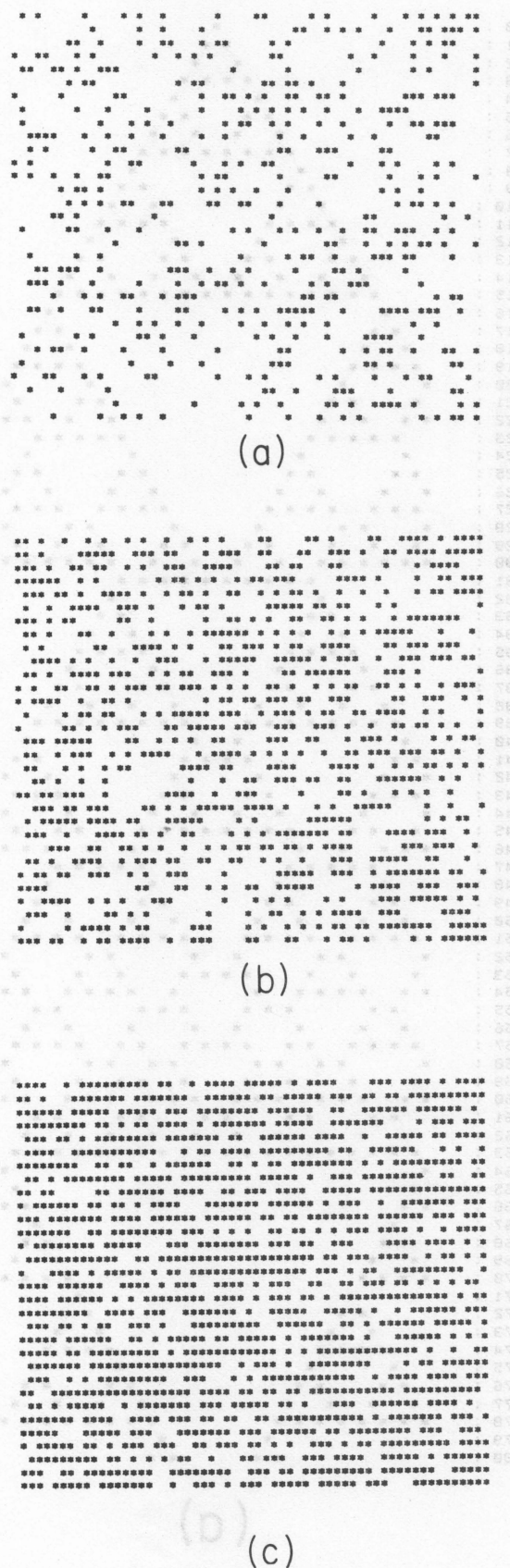

(a)

(b)

(c)

FIG. 10. Examples of sets of disordered configurations in which each site is chosen to have value 1 with independent probability (a) 0.25, (b) 0.5, and (c) 0.75. Successive lines are independent. The configurations are to be compared with those generated by cellular automaton evolution as shown in Fig. 8.
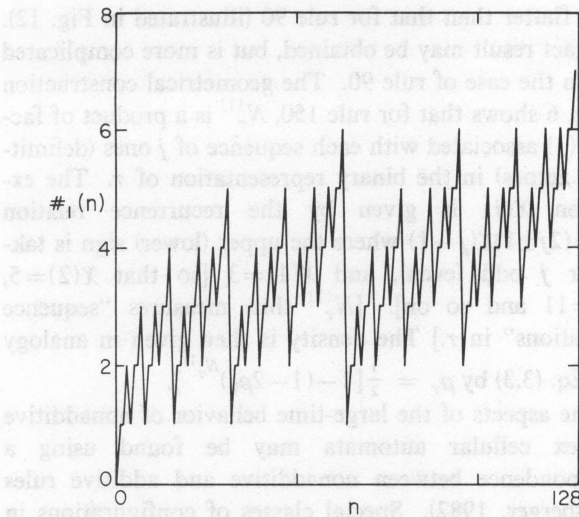
FIG. 11. The number of occurrences $\#_1(n)$ of the binary digit 1 in the binary representation of the integer n [$\#_1(1)=1$, $\#_1(2)=1$, $\#_1(3)=2$, $\#_1(4)=1$, and so on]. The function is defined only for integer n: values obtained for successive integer n have nevertheless been joined by straight lines.

transients may occur, but the probability that a nonzero value survives at a particular site for τ time steps assuming an initial disordered state with $\rho=\frac{1}{2}$ is $2^{-3(2\tau+1)}$. Rule 254 yields $\rho_\infty=1$, with a probability $(1-\rho_0)^{2\tau+1}$ for a transient of length $\geq\tau$. Rule 204 is the "identity rule," which propagates any initial configuration unchanged and yields $\rho_\infty=\rho_0$. The "disjunctive superposition" principle for rule 250 discussed in Sec. II implies $\rho_\infty=1$. For rule 50, the "conjunctive superposition" principle yields $\rho_\infty=\frac{1}{2}$.

Other simple rules serve as "filters" for specific initial sequences, yielding final densities proportional to the initial density of the sequences to be selected. For rule 4, the final density is equal to the initial density of 101 sequences, so that $\rho_\infty=\rho_0^2(1-\rho_0)$. For rule 36, ρ_∞ is determined by the density of initial 00100 and ...1010101... sequences and is approximately $\frac{1}{16}$ for $\rho_0=\frac{1}{2}$.

Exact results for the behavior of ρ_τ with the modulo-two rule 90 may be derived using the additive superposition property discussed in Sec. II.

Consider first the number of sites $N_\tau^{(1)}$ with value 1 obtained by evolution according to rule 90 from an initial state containing a single site with value 1, as illustrated in Fig. 3. Geometrical considerations based on Fig. 5 yield the result[6]

$$N_\tau^{(1)} = 2^{\#_1(\tau)}, \qquad (3.2)$$

where the function $\#_1(\tau)$ gives the number of occurrences of the digit 1 in the binary representation of the integer τ, as defined above, and is illustrated in Fig. 11. Equation (3.2) may be derived as follows. Consider the

figure generated by $\lceil\log_2\tau\rceil$ (the number of digits in the binary representation of τ) steps in the construction of Fig. 5. The configuration obtained after τ time steps of cellular automaton evolution corresponds to a slice through this figure, with a 1 at each point crossed by a line of the figure, and 0 elsewhere. By construction, the slice must lie in the lower half of the figure. Successive digits in the binary representation of τ determine whether the slice crosses the upper (0) or lower (1) halves of successively smaller triangles. The number of lines of the figure crossed is multiplied by a factor each time the lower half is chosen. The total number of sites with value 1 encountered is then given by a product of the factors of two associated with each 1 digit in the binary representation of τ. Inspection of Fig. 5 also yields a formula for the positions of all sites with value 1. With the original site at position 0, the positions of sites with value 1 after τ time steps are given by $\pm(2^{j_1}\pm(2^{j_2}\pm\cdots))$, where all possible combinations of signs are to be taken, and the j_i correspond to the positions at which the digit 1 appears in the binary representation of τ, defined so that $\tau = 2^{j_1} + 2^{j_2} + \cdots$ and $j_1>j_2>\cdots$.

Equation (3.2) shows that the density averaged over the region of nonzero sites ("light cone") in the rule 90 evolution of Fig. 3 is given by $\rho_\tau=N_\tau^{(1)}/(2\tau+1)$ and does not tend to a definite limit for large τ. Nevertheless, the time-average density

$$\bar{\rho}_T=(1/T)\sum_{\tau=0}^{\tau=T}\rho_\tau$$

tends to zero (as expected from the geometrical construction of Fig. 5) like $T^{\log_2 3-2}\sim T^{-0.42}$.[7] Results for initial states containing a finite number of sites with value 1 may be obtained by additive superposition. If the initial configuration is one which would be reached by evolution from a single site after, say, τ_0 time steps, then the resulting density is given by Eq. (3.2) with the replacement $\tau\rightarrow\tau-\tau_0$. Only a very small fraction of initial configurations may be treated in this way, since evolution from a single site generates only one of the 2^k possible configurations in which the maximum separation between nonzero sites is k. For small or highly regular initial configurations, results analogous to (3.2) may nevertheless be derived. Statistical results for evolution from disordered initial states may also be derived. Equation (3.2) implies that after exactly $\tau=2^j$ time steps, an initial state containing a single nonzero site evolves to a configuration with only two nonzero sites. At this point, the value of a particular site at position n is simply the sum modulo two of the initial values of sites at positions $n-\tau$ and $n+\tau$. If we start from a disordered initial configuration, the density at such time steps is thus given by $\rho_{\tau=2^j}=2\rho_0(1-\rho_0)$.

---

[6] This result has also been derived by somewhat lengthy algebraic means in Glaisher (1899), Fine (1947), Roberts (1957), Kimball et al. (1958), and Honsberger (1976).

[7] This form is strictly correct only for $T=2^k$. For $T=2^k(1+\delta)$, there is a correction factor $\simeq(1+\delta^{\log_2 3})/(1+\delta)^{\log_2 3}$, which lies between 0.86 and 1, with a broad minimum around $\delta=0.3$.

In general, the value of a site at time step τ is a sum modulo two of the initial values of $N_\tau^{(1)} = 2^{\#_1(\tau)}$ sites, which each have value 1 with probability ρ_0. If each of a set of k sites has value 1 with probability p, then the probability that the sum of the values at the sites will be odd (equal to 1 modulo two) is

$$\sum_{i \text{ odd}} \binom{k}{i} p^i (1-p)^{k-i} = \frac{1}{2}[1-(1-2p)^k] .$$

Thus the density of sites with value 1 obtained by evolution for τ time steps from an initial state with density ρ_0 according to cellular automaton rule 90 is given by

$$\rho_\tau = \frac{1}{2}[1-(1-2\rho_0)^{2^{\#_1(\tau)}}] . \tag{3.3}$$

This result is shown as a function of τ for the case $\rho_0 = 0.2$ in Fig. 12. For large τ, $\#_1(\tau) = O(\log_2\tau)$, except at a set of points of measure zero, and Eq. (3.3) implies that $\rho_\tau \to \frac{1}{2}$ as $\tau \to \infty$ for almost all τ (so long as $\rho_0 \neq 0$).

Cellular automaton rule 150 shares with rule 90 the property of additive superposition. Inspection of the results for rule 150 given in Fig. 3 indicates that the value of a particular site depends on the values of at least three initial sites (this minimum again being achieved when $\tau = 2^k$), so that $|\rho_\tau - \frac{1}{2}| \leq |1-2\rho_0|^3$. Between the exceptional time steps $\tau = 2^k$, the ρ_τ for rule 150 tends to be
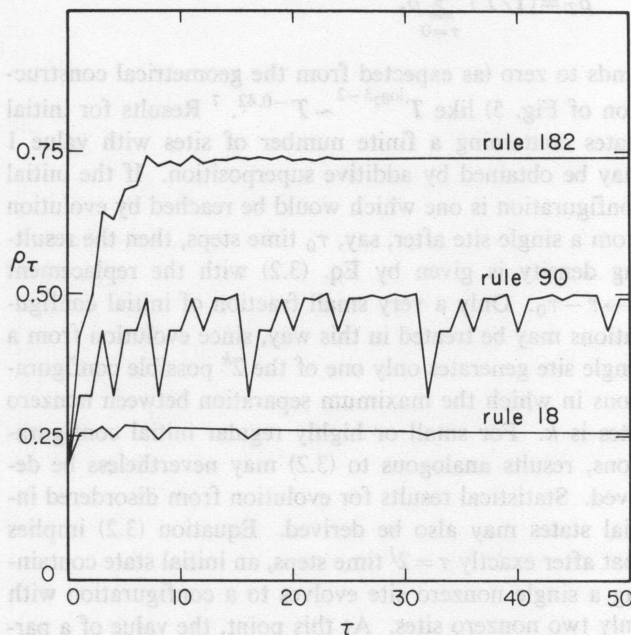

FIG. 12. Average density ρ_τ of sites with value 1 obtained by time evolution according to various cellular automaton rules starting from a disordered initial state with $\rho_0 = 0.2$. The additivity of the modulo-two rule 90 may be used to derive the exact result (3.2) for ρ_τ. The irregularities appear for time steps at which the value of each site depends on the values of only a few initial sites. For the nonadditive complex rules exemplified by 18 and 182, the values of sites at time step τ depend on the values of $O(\tau)$ initial sites, and ρ_τ tends smoothly to a definite limit. This limit is independent of the density of the initial disordered state.

much flatter than that for rule 90 (illustrated in Fig. 12). An exact result may be obtained, but is more complicated than in the case of rule 90. The geometrical construction of Fig. 6 shows that for rule 150, $N_\tau^{(1)}$ is a product of factors $X(j)$ associated with each sequence of j ones (delimited by zeroes) in the binary representation of τ. The expression $X(j)$ is given by the recurrence relation $X(j) = (2j \pm 1)X(j-1)$ where the upper (lower) sign is taken for j odd (even), and $X(1) = 3$ [so that $X(2) = 5$, $X(3) = 11$ and so on]. [$N_\tau^{(1)}$ thus measures "sequence correlations" in τ.] The density is then given in analogy with Eq. (3.3) by $\rho_\tau = \frac{1}{2}[1-(1-2\rho_0)^{N_\tau^{(1)}}]$.

Some aspects of the large-time behavior of nonadditive complex cellular automata may be found using a correspondence between nonadditive and additive rules (Grassberger, 1982). Special classes of configurations in nonadditive cellular automata effectively evolve according to additive rules. For example, with the nonadditive complex rule 18, a configuration in which, say, all even-numbered sites have value zero evolves after one time step to a configuration with all odd-numbered sites zero, and with the values of even-numbered sites given by the sums modulo two of their odd-numbered neighbors on the previous time step, just as for the additive rule 90. An arbitrary initial configuration may always be decomposed into a sequence of (perhaps small) "domains," in each of which either all even-numbered sites or all odd-numbered sites have value zero. These domains are then separated by "domain walls" or "kinks." The kinks move in the cellular automaton evolution and may annihilate in pairs. The motion of the kinks is determined by the initial configuration; with a disordered initial configuration, the kinks initially follow approximately a random walk, so that their mean displacement increases with time according to $\langle x^2 \rangle = t$ (Grassberger, 1982), and the paths of the kinks are fractal curves. This implies that the average kink density decreases through annihilation as if by diffusion processes according to the formula $\langle \rho_{\text{kink}} \rangle \sim (4\pi t)^{-1/2}$ (Grassberger, 1982). Thus after a sufficiently long time all kinks (at least from any finite initial configuration) must annihilate, leaving a configuration whose alternate sites evolve according to the additive cellular automaton rule 90. Each point on the "front" formed by the kink paths yields a pattern analogous to Fig. 5. The superposition of such patterns, each diluted by the insertion of alternate zero sites, yields configurations with an average density $\frac{1}{4}$ (Grassberger, 1982). The large number of sites on the "front" suppresses the fluctuations found for complete evolution according to additive rule 90. Starting with a disordered configuration of any nonzero density, evolution according to cellular automaton rule 18 therefore yields an asymptotic density $\frac{1}{4}$. The existence of a universal ρ_∞, independent of initial density ρ_0, is characteristic of complex cellular automaton rules.

Straightforward transformations on the case of rule 18 above then yield asymptotic densities $\rho_\infty = \frac{1}{4}$ for the complex nonadditive rules 146, 122, and 126, and an asymptotic density $\frac{3}{4}$ for rule 182, again all independent of the initial density ρ_0 (Grassberger, 1982). No simple domain

structure appears with rule 22, and the approach fails. Simulations yield a numerical estimate $\rho_\infty = 0.35 \pm 0.02$ for evolution from disordered configurations with any nonzero ρ_0.

Figure 12 shows the behavior of ρ_τ for the complex nonadditive cellular automata 18 and 182 with $\rho_0 = 0.2$, and suggests that the final constant values $\rho_\infty = 0.25$ and $\rho_\infty = 0.75$ are approached roughly exponentially with time.

One may compare exact results for limiting densities of cellular automata with approximations obtained from a statistical approach (akin to "mean-field theory"). As discussed above, cellular automaton evolution generates correlations between values at different sites. Nevertheless, as a simple approximation, one may ignore these correlations, and parametrize all configurations by their average density ρ, or, equivalently, by the probabilities p and $q = 1 - p$, assumed independent, for each site to have value 1 and 0, respectively. With this approximation, the time evolution of the density is given by a master equation

$$\frac{\delta\rho}{\delta\tau} = \Gamma(0 \rightarrow 1) - \Gamma(1 \rightarrow 0) ,$$

$$\Gamma(0 \rightarrow 1) = \mathbf{P} \cdot (00110011 \wedge \mathbf{R}) ,$$

$$\Gamma(1 \rightarrow 0) = \mathbf{P} \cdot (11001100 \wedge \sim \mathbf{R}) , \quad (3.4)$$

$$\mathbf{P} = \{ p^3, p^2 q, p^2 q, pq^2, p^2 q, pq^2, pq^2, q^3 \} .$$

The term $\Gamma(0 \rightarrow 1)$ represents the average fraction of sites whose values change from 0 to 1 in each time step, and $\Gamma(1 \rightarrow 0)$ the fraction changing from 1 to 0. $\mathbf{R}$ is the binary specification of a cellular automaton rule, and the binary number with which it is "masked" (digitwise conjunction) selects local rules for three-site neighborhoods with appropriate values at the center site. $\mathbf{P}$ is the vector of probabilities for the possible three-site neighborhoods, assuming each site independently to have value 1 with probability $p = \rho$, and to have value 0 with probability $q = 1 - p = 1 - \rho$. The dot indicates that each element of this vector is to be multiplied by the corresponding digit of the binary sequence, and the results are to be added together. The equilibrium density ρ_∞ is achieved when

$$\frac{\delta\rho}{\delta\tau} = 0.$$

This condition yields a polynomial equation for p and thus ρ_∞ for each of the legal cellular automaton rules. For rule 90, the equation is $pq^2 - p^3 = p - 2p^2 = p(1 - 2p) = 0$, which has solutions $p = 0$ (null state for all time) and $p = \frac{1}{2}$. Rule 18 yields the equation $pq^2 - 2p^2 q - p^3 = p(1 - 4p + 2p^2) = 0$, which has the solutions $p = 0$ and $p = 1 - 1/\sqrt{2} \simeq 0.293$, together with the irrelevant solution $p = 1 + 1/\sqrt{2} > 1$. Rule 182 yields $2pq^2 - p^2 q = p(2 - 3p)(1 - p) = 0$, giving $p = 0, 1, \frac{2}{3}$. For rules 90 and 18, these approximate results are close to the exact results 0.5 and 0.25. For rule 182, there is a significant discrepancy from the exact value 0.75. Nevertheless, for all complex cellular automaton rules, it appears

that the master equation (3.4) yields equilibrium densities within 10–20% of the exact values. The discrepancies are a reflection of the violation of the Markovian approximation required to derive Eq. (3.4) and thus of the presence of correlations induced by cellular automaton evolution.

In the discussion above, a definite value for the density ρ_τ at each time step was found by averaging over all sites of an infinite cellular automaton. If instead the density is estimated by averaging over blocks containing a finite number of sites b, a distribution of density values is obtained. In a disordered state, the central limit theorem ensures that for large b, these density estimates follow a Gaussian distribution with standard deviation $\simeq 1/\sqrt{b}$. Evolution according to any of the complex cellular automaton rules appears accurately to maintain this Gaussian distribution, while shifting its mean as illustrated in Fig. 12. Density in cellular automaton configurations thus obeys the "law of large numbers." Instead of taking many blocks of sites at a single time step, one might estimate the density at "equilibrium" by averaging results for a single block over many time steps. For nonadditive complex cellular automaton rules, it appears that these two procedures yield the same limiting results. However, the large fluctuations in average density visible in Fig. 12 at particular time steps for additive rules (90 and 150) would be lost in a time average.

Cellular automaton evolution is supposed to generate correlations between values at different sites. The very simplest measure of these correlations is the two-point correlation function $C^{(2)}(r) = \langle S(m)S(m+r) \rangle - \langle S(m) \rangle \times \langle S(m+r) \rangle$, where the average is taken over all possible positions m in the cellular automaton at a fixed time, and $S(k)$ takes on values -1 and $+1$ when the site at position k has values 0 and 1, respectively. A disordered configuration involves no correlations between values at different sites and thus gives $C^{(2)}(r) = 0$ for $r > 0$ [$C^{(2)}(0) = 1 - (2\rho - 1)^2$]. With the single-site initial state of Fig. 3, evolution of complex cellular automata yields configurations with definite periodicities. These periodicities give rise to peaks in $C^{(2)}(r)$. At time step τ, the largest peaks occur when $r = 2^k$ and the digit corresponding to 2^k appears in the binary decomposition of τ; smaller peaks occur when $r = 2^{k_1} \pm 2^{k_2}$, and so on. For the additive cellular automaton rules 90 and 150, a convolution of this result with the correlation function for any initial state gives the form of $C^{(2)}(r)$ after evolution for τ time steps. With these rules, the correlation function obtained by evolution from a disordered initial configuration thus always remains zero. For nonadditive rules, nonzero short-range correlations may nevertheless be generated from disordered initial configurations. The form of $C^{(2)}(r)$ for rule 18 at large times is shown in Fig. 13, and is seen to fall roughly exponentially with a correlation length ~ 2. The existence of a nonzero correlation length in this case is our first indication of the generation of order by cellular automaton evolution.

Figures 3 and 8 show that the evolution of complex cellular automata generates complicated patterns with a dis-

tinctive structure. The average density and the two-point correlation function are too coarse as statistical measures to be sensitive to this structure. Individual configurations appear to contain long sequences of correlated sites, punctuated by disordered regions. The two-dimensional picture formed by the succession of configurations in time is characteristically peppered with triangle structures. These triangles are formed when a long sequence of sites which suddenly all attain the same value, as if by a fluctuation, is progressively reduced in length by "ambient noise." Let $T_{(i)}(n)$ denote the density of triangles (in position and time) with base length n and filled with sites of value i. It is convenient to begin by considering the behavior of this density and then to discuss its consequences for the properties of individual configurations, whose long sequences typically correspond to sections through the triangles.

Consider first evolution from a simple initial state containing a single site with value 1. Figure 3 shows that in this case, all complex cellular automata (except rule 150) generate a qualitatively similar pattern, containing many congruent triangles whose bases have lengths 2^k. A geometrical construction for the limiting pattern obtained at large times was given in Fig. 5. At each successive stage in the construction, the linear dimensions (base lengths) of the triangles added are halved, and their number is multiplied by a factor 3. In the limit, therefore, $T(n/2) \sim 3T(n)$, (with $n = 2^k$), and hence

$$T(n) \sim n^{-\log_2 3} \sim n^{-1.59} \tag{3.5}$$

[requiring exactly one triangle of size $\tau/2$ at time step τ

fixes the normalization as $T(n) = (2n/\tau)^{-\log_2 3}$]. The result (3.5) demonstrates that the patterns obtained from complex cellular automata in Fig. 3 not only contain structure on all scales (in the form of triangles of all sizes), but also exhibit a scale invariance or self similarity which implies the same structure on all scales (cf. Mandelbrot, 1982; Willson, 1982). The power law form of the triangle density (3.5) is independent of the absolute scale of n.

Self-similar figures on, for example, a plane may in general be characterized as follows. Find the minimum number $N(a)$ of squares with side a necessary to cover all parts of the figure (all sites with nonzero values in the cellular automaton case). The figure is self-similar or scale invariant if rescaling a changes $N(a)$ by a constant factor independent of the absolute size of a. In this case, $N(a) \sim a^{-D}$, where D is defined to be the Hausdorff-Besicovitch or fractal dimension (Mandelbrot, 1977, 1982) of the figure. A figure filling the plane would give $D = 2$, while a line would give $D = 1$. Intermediate values of D indicate clustering or intermittency. According to this definition, the cellular automaton pattern of Fig. 5 has fractal dimension $D = \log_2 3 \simeq 1.59$.

Figure 6 gives the construction analogous to Fig. 5 for the pattern generated by rule 150 in Fig. 3. In this case, the triangle density satisfies the two-term recurrence relation $T(n = 2^k) = 2T(2^{k+1}) + 4T(2^{k+2})$ with, say, $T(1) = 0$ and $T(2) = 2$. For large k, this yields (in analogy with the Fibonacci series)[8]

$$T(n) \sim n^{-\log_2(2\varphi)} = n^{-\log_2(1+\sqrt{5})} \sim n^{1.69}, \tag{3.6}$$

where $\varphi = (1 + \sqrt{5})/2 \simeq 1.618$ is the "golden ratio" which solves the equation $x^2 = x + 1$. The limiting fractal dimension of the pattern in Fig. 6 generated by cellular automaton rule 150 is thus $\log_2(2\varphi) = 1 + \log_2(\varphi) \simeq 1.69$.

The self similarity of the patterns generated by time evolution with complex cellular automaton rules in Fig. 3 is shared by almost all the configurations appearing at particular time steps and corresponding to lines through the patterns. If the fractal dimension of the two-dimensional patterns is D, then the fractal dimension of almost all the individual configurations is $D - 1$. The configurations obtained at, for example, time steps τ of the form 2^k are members of an exceptional set of measure zero, for which no fractal dimension is defined. Almost all configurations generated from a single initial site by complex cellular automaton rules are thus self-similar, and (except for rule 150) are characterized by a fractal di-

FIG. 13. Two-point correlation function $C^{(2)}(r)$ for configurations generated at large times by evolution according to cellular automaton rule 18 from any disordered initial configuration. $C^{(2)}(r)$ is defined as $\langle S(m)S(m+r)\rangle - \langle S(m)\rangle\langle S(m+r)\rangle$, where the average is taken over all sites m of the cellular automaton, and $S(k) = \pm 1$ when site k has values 1 and 0, respectively. No correlations are present in a disordered configuration, so that $C^{(2)}(r) = 0$ for $r > 0$. Evolution according to certain complex cellular automaton rules, such as 18, yield nonzero but exponentially damped correlations.

---

[8] For small k, the triangle density in this case does not behave as a pure power of 2^k. Whereas the solution to any one-term recurrence relation, of the type found for cellular automaton rule 90, is a pure power, the solution to a p-term recurrence relation is in general a sum of p powers, with each exponent given by a root of the characteristic polynomial equation. In the high-order limit, the solutions are dominated by the term with the highest exponent (corresponding to the largest root of the equation). Complex roots yield oscillatory behavior [as in $f(k) = -f(k-1) + f(k-2); f(0) = 0, f(1) = 1$].

mension $D = \log_2 3 - 1 = \log_2(\frac{3}{2}) \simeq 0.59$. The second form may be deduced directly from the geometrical construction of Fig. 5. For rule 150, the configurations have fractal dimension $D = \log_2 \varphi$.

Figure 14 shows patterns generated by evolution with a selection of complex cellular automaton rules from initial states containing a few sites with value 1, extending over a region of size n_0. Comparison with Fig. 3 demonstrates that in most cases the patterns obtained even after many time steps differ from those generated with a single initial site. A few exceptional initial configurations (such as the one used for the first rule 90 example in Fig. 14) coincide with configurations reached by evolution from a single initial site and therefore yield a similar pattern, appropriately shifted in time. In the general case, Fig. 14 suggests that the form of the initial state determines the number of triangles with size $n \lesssim n_0$, but does not affect the density of triangles with $n \gg n_0$. As a simple example consider the modulo-two rule 90, whose additive superposition property implies that the final pattern obtained from an arbitrary initial state is simply a superposition of the patterns which would be generated from each of the nonzero initial sites in isolation. These latter patterns were shown in Fig. 5, and involve the generation of a triangle of size 2^k at time step 2^k. The superposition of such patterns yields at time step 2^k a triangle of size at least $2^k - 2n_0$. This conclusion apparently holds also for nonadditive complex cellular automata, so that, in general, for $n \gg n_0$, the density of triangles follows the form (3.5), as for a single site initial state. The patterns thus exhibit self-similarity for features large compared to the intrinsic scale defined by the "size" of the initial state. One therefore concludes that patterns which "grow" from any simple initial state according to any of the "complex" cellular automaton rules (except 150) share the universal feature of self similarity, characterized by a fractal dimension $\log_2 3$. On this basis, one may then conjecture that given suitable geometry (perhaps in more than one dimension, and possibly with more than three sites in a neighborhood), many of the wide variety of systems found to exhibit self-similar structure (Mandelbrot, 1977, 1982) attain this structure through local processes which follow cellular automaton rules.

Having considered the case of simple initial configurations, we now turn to the case of evolution from disordered initial configurations, illustrated in Fig. 8. Figure 15 shows the first 300 time steps in the evolution of cellular automaton 126, starting from a disordered initial state with density $\rho = 0.5$. Triangles of all sizes appear to be generated (the largest appearing in the figure has $n = 27$). Figure 16 shows the density of triangles $T(n)$ obtained at large times by evolution according to rule 126 and all of the other complex cellular automaton rules. The figure reveals the remarkable fact that for large n, all nonadditive rules yield the same $T(n)$, distinct from that for the additive rules (90 and 150). All the results are well fit by the form

$$T(n) \sim \lambda^{-n} . \tag{3.7}$$

For nonadditive rules $\lambda \sim \frac{4}{3}$, while for the additive rules $\lambda \sim 2$. The same results are obtained at large times regardless of the density of the initial state. Thus the spectrum of triangles generated by complex cellular automaton evolution is universal, independent both of the details of the initial state, and of the precise cellular automaton rule used.

The behavior (3.8) of the triangle density with disordered initial states is to be contrasted with that of (3.5) for simple initial states. The precise form of an initial state of finite extent n_0 affects the pattern generated only at length scales $\lesssim n_0$: at larger length scales the pattern takes on a universal self-similar character. A disordered initial state of infinite extent affects the pattern generated at all length scales and for all times. Triangles of all sizes are nevertheless obtained, so that structure is generated on all scales, as suggested by Fig. 15. However, the pattern is not self-similar, but depends on the absolute scale defined by the spacing between sites.

Disordered configurations are defined to involve no statistical correlations between values at different sites. They thus correspond to a discrete form of white noise and yield a flat spatial frequency spectrum. One may also consider "pseudodisordered" configurations in which the value of each individual site is chosen randomly, but according to a distribution which yields statistical correlations between different sites, and a nontrivial spatial Fourier spectrum. For example, a Brownian configuration (with spatial frequency spectrum $1/k^2$) is obtained by assigning a value to each site in succession, with a certain probability for the value to differ from one site to the next (as in a random walk). The patterns generated by cellular automaton from such initial configurations may differ from those obtained with disordered (white noise) initial configurations. Complex nonadditive cellular automata evolving from a Brownian initial state yield patterns whose triangle density $T(n)$ decreases less rapidly at large n than for disordered initial configurations: the "long-range order" of the initial state leads to the generation of longer-range fluctuations. In the extreme limit of a homogeneous initial state (such as $\ldots 11111 \ldots$ or $\ldots 10101 \ldots$), cellular automaton evolution preserves the homogeneity, and no finite structures are generated.

The appearance of triangles over a series of time steps in the evolution of complex cellular automata from disordered initial states reflects the generation of long sequences of correlated sites in individual cellular automaton configurations. This effect is measured by the "sequence density" $Q_{(i)}(n)$, defined as the density of sequences of exactly n adjacent sites with the same value i (bordered by sites with a different value). Thus, for example, $Q_{(0)}(4)$ gives the density of 100001 sequences. $Q_{(0)}(n)$ clearly satisfies the sum rule

$$\sum_{n=1}^{\infty} n Q_{(0)}(n) = 1 - \rho .$$

In a disordered configuration with density $p = 1 - q$, $Q_{(0)}(n) \sim p^2 q^n$ for large n. Any sequence longer than two sites in a complex cellular automaton must yield a trian-

RULE : 01011010 (90)   RULE : 01111110 (126)   RULE : 11011010 (218)
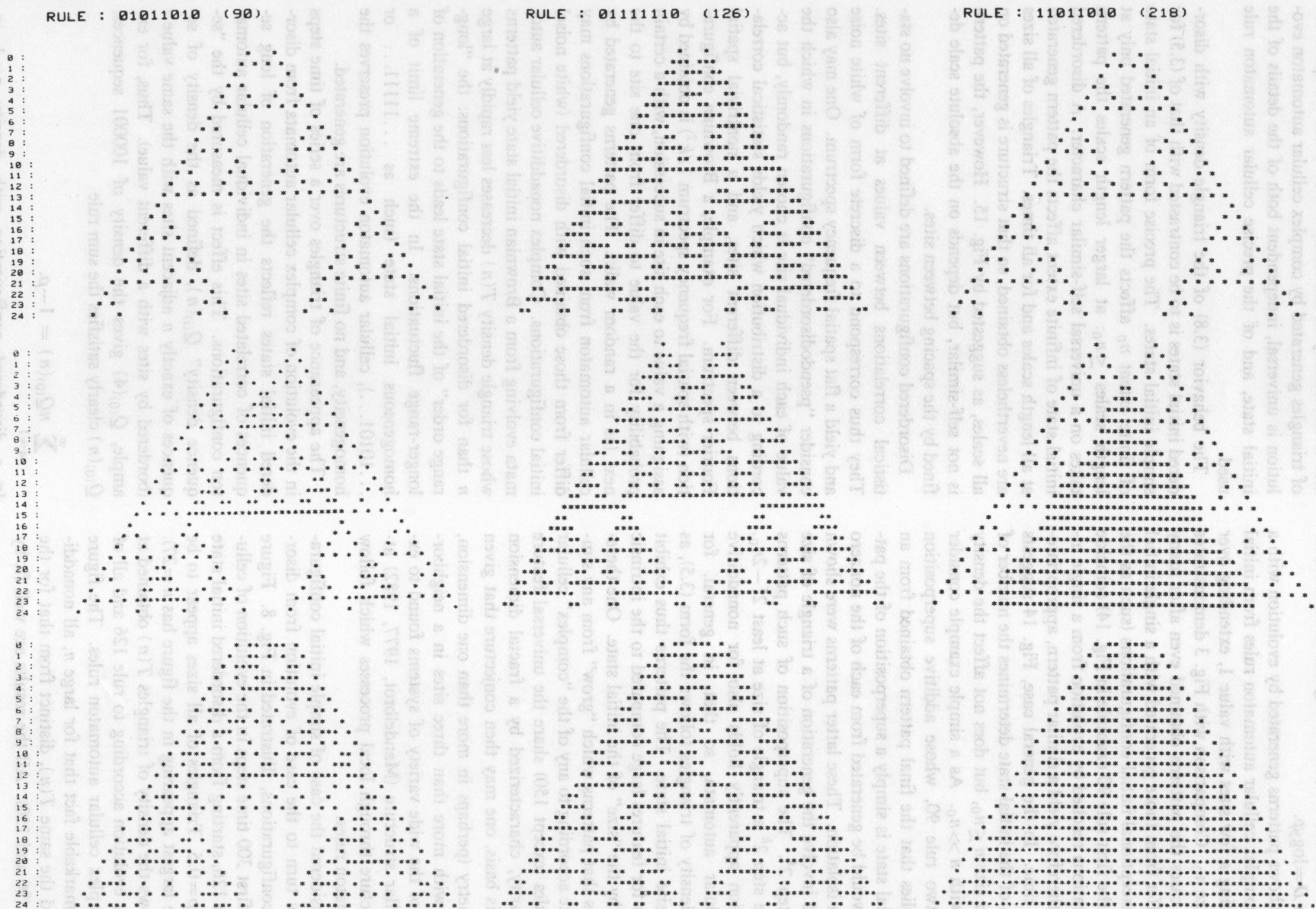

FIG. 14. Twenty-five time steps in the evolution of several simple initial configurations according to cellular automaton rules 90, 126, and 218. Configurations generated by rule 90 obey additive superposition (under addition modulo two). The first initial state taken is exceptional for rules 90 and 218, since it occurs in evolution from a single initial site, as shown in Fig. 3, so that the final pattern is a shifted form of that found in Fig. 3. For other initial states, the patterns obtained deviate substantially from those of Fig. 3. However, features with sizes much larger than the extent of the initial state remain unchanged. For complex cellular automaton rules such as 90 and 126, such features share the self-similarity found in Fig. 3.

FIG. 16. Density $T(n)$ of triangle structures generated in the evolution of all the possible complex cellular automata from disordered initial configurations with density $\rho_0=0.5$. Triangles are evident in Figs. 8 and 16. They are formed when a sequence of sites suddenly attain the same value, but the length of the sequence is progressively reduced on subsequent time steps, until the apex of the triangle is reached. The appearance of triangles is a simple indication of self-organization. The triangle density $T(n)$ is defined only at integer values of n, but these points have been joined in the figure. For large n, the triangle densities for all complex cellular automata are seen to tend towards one of two limiting forms. The group tending to the upper curve are the nonadditive complex cellular automata 18, 22, 122, 126, 146, and 182. The additive rules 90 and 150 follow the lower curve. In both cases, $T(n)$ falls off exponentially with n, in contrast to the power law form found for the self-similar patterns of Figs. 3, 5, and 14.

FIG. 15. Configurations obtained by evolution for 300 time steps from an initial disordered configuration with $\rho=0.5$ according to cellular automaton rule 126. The fluctuations visible in the form of triangles and apparent at small scales in Fig. 8 are seen here to occur on all scales. The largest triangle in this sample has a base length of 27 sites.

gle, leading to the sum rule

$$Q(n)\simeq\sum_{i=n}^{\infty}[2T(i)/i].$$

Thus the $Q(n)$ obtained at large times by evolution from a disordered initial state should follow the same exponential form (3.8) as $T(n)$.

Figure 17 shows the sequence density $Q_{(0)}(n)$ obtained at various time steps in the evolution of rule 126 from a

disordered initial state, as illustrated in Fig. 15. At each time step, the $Q_{(0)}(n)$ for a disordered configuration (illustrated in Fig. 10) with the same average density has been subtracted. The resulting difference vanishes by definition at $\tau=0$, but Fig. 17 shows that for $\tau\geq1$, the cellular automaton evolution yields a nonzero difference. After a few time steps, the cellular automaton tends to an equilibrium state containing an excess of long sequences of sites with value 0, and a deficit of short ones. This final equilibrium $Q_{(0)}(n)$ does not depend on the density of the initial disordered configuration. Starting from any disordered initial state (random noise), repeated application of the local cellular automaton rules thus generates ordered configurations whose statistical properties, as measured by sequence densities, differ from those of corresponding disordered configurations. The impression of self-organization in individual configurations given by Fig. 8 is thus quantitatively confirmed.

As suggested by the sum rule, the $Q_{(0)}(n)$ for complex cellular automata with disordered initial states follow the exponential behavior (3.7) found for the $T(n)$. Again, the parameter λ has a universal value $\sim\frac{4}{3}$ for all nonadditive cellular automaton rules and ~2 for additive ones. If all configurations of the cellular automata were disordered, then the sequence density would behave at large n as
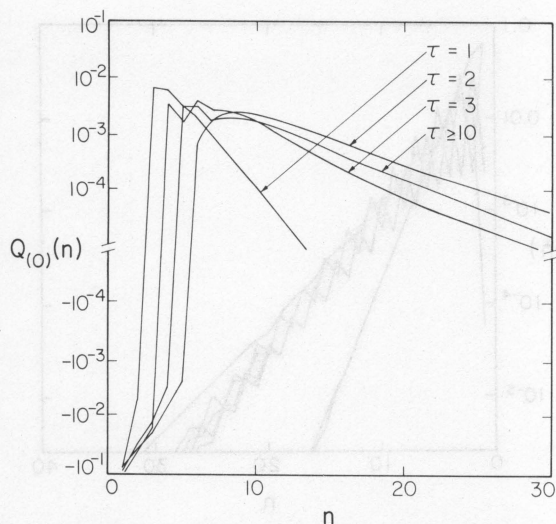
FIG. 17. Density $Q_{(0)}(n)$ of sequences of exactly n successive sites with value 0 (delimited by sites with value 1) in configurations generated by τ steps in time evolution according to cellular automaton rule 126, starting from an initial disordered state with density $\rho = 0.5$. [The function $Q_{(0)}(n)$ is defined only for integer n: points are joined for ease of identification.] At each time step, the density of sequences in a disordered configuration with the same average total density has been subtracted. This difference vanishes for $\tau = 0$ by definition. The nonzero value shown in the figure for $\tau \geq 1$ is a manifestation of self-organization in the cellular automaton, suggested qualitatively by comparison of Figs. 8 and 10. For large τ, an equilibrium state is reached, which exhibits an excess of long sequences and a deficit of short ones.

$(1-\rho)^n$ and depend on total average density ρ for the configurations. The form (3.5) yields sequence correlations with the same exponential behavior, but with a fixed λ, universal to all the nonadditive complex cellular automaton rules, and irrespective of the final densities to which they lead. (The universal form may be viewed as corresponding to an "effective density" $\simeq 0.25$.)

Cellular automata are usually defined to evolve according to definite deterministic local rules. In modelling physical or biological systems it is, however, sometimes convenient to consider cellular automata whose local rules involve probabilistic elements or noise (cf. Griffeath, 1970; Schulman and Seiden, 1978; Gach et al., 1978). The simplest procedure is to prescribe that at each time step the value obtained by application of the deterministic rule at each site is to be reversed with a probability κ (and with each site treated independently). (If an energy is associated with the reversal of a site, κ gives the Boltzmann factor corresponding to a finite temperature heat bath.) Figure 18 shows the effects of introducing such noise in the evolution of cellular automaton rule 126. The structures generated are progressively destroyed as κ increases. Investigation of densities and correlation functions indicates that the transition to disorder is a continuous one, and no phenomenon analogous to a "phase transition" is found.

FIG. 18. Configurations generated from a disordered initial state (with $\rho_0 = 0.5$) by the evolution of the complex nonadditive cellular automaton 126, in the presence of noise which causes values obtained at each site to be reversed with probability κ at every time step. (a) is for $\kappa = 0$ (no "noise"), (b) for $\kappa = 0.1$, (c) for $\kappa = 0.2$, and (d) for $\kappa = 0.5$. As κ increases, the structure generated is progressively destroyed. No discontinuity in behavior as a function of κ is found.

IV. GLOBAL PROPERTIES OF ELEMENTARY CELLULAR AUTOMATA

Section III analyzed the behavior of cellular automata by considering the statistical properties of the set of values of sites in individual cellular automaton configurations. The alternative approach taken in this section considers the statistical properties of the set (ensemble) comprising all possible complete configurations of a cellular automaton (in analogy with the Γ-space approach to classical statistical mechanics). Such an approach provides connections with dynamical systems theory (Ott, 1981) and the formal theory of computation (Minsky, 1967; Arbib, 1969; Manna, 1974; Hopcroft and Ullman, 1979; Beckman, 1980), and yields a view of self-organization phenomena complementary to that developed in Sec. III. Cellular automaton rules may be considered as a form of "symbolic dynamics" (e.g., Alekseev and Yakobson, 1981), in which the degrees of freedom in the system are genuinely discrete, rather than being continuous but assigned to discrete "bins."

As in Sec. III, we examine here only elementary cellular automata. Some results on global properties of more complicated cellular automata will be mentioned in Sec. V.

For most of this section, it will be convenient to consider "finite" cellular automata, containing only a finite number of sites N. There are a total of 2^N possible configurations for such a cellular automaton. Each configuration is uniquely specified by a length N binary integer whose digits give the values of the corresponding sites.[9] (A configuration of an infinite cellular automaton would correspond to a binary real number.) The evolution of a finite cellular automaton depends on the boundary conditions applied. We shall usually assume periodic boundary conditions, in which the first and last sites are identified, as if the sites lay on a circle of circumference N. One could alternatively take an infinite sequence of sites, but assume that all those outside the region of length N have value 0. Results obtained with these two choices were compared in Fig. 9, and no important qualitative differences were found. Most of the results derived in this section are also insensitive to the form of boundary conditions assumed. However, several of the later ones depend sensitively on the value of N taken.

Cellular automaton rules define a transformation from one sequence of binary digits to another. The rules thus provide a mapping from the set of binary numbers of length N onto itself. For the trivial case of rule 0, all binary numbers are mapped to zero. Figure 19 shows the mappings corresponding to evolution for one and five
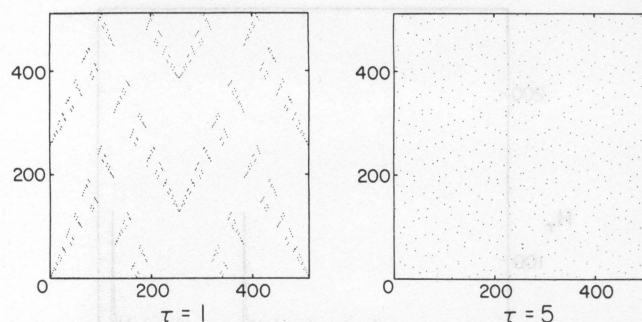
FIG. 19. Mapping in the set of 512 possible configurations of a length nine finite cellular automaton corresponding to evolution for τ time steps according to the modulo-two rule 90. Each possible configuration is represented by the decimal equivalent of the binary number whose digits give the values at each of its sites. The horizontal axis gives the number specifying the initial configuration; the vertical axes that for the final configuration. Each initial configuration is mapped to a unique final configuration.

time steps according to cellular automaton rule 90 with $N=9$. The mapping corresponding to one time step is seen to maintain some nearby sets of configurations. After five time steps, however, the evolution is seen to map configurations roughly uniformly, so that the final configurations obtained from nearby initial configurations are essentially uncorrelated.

A convenient measure of distance in the space of cellular automaton configurations is the "Hamming distance" $H(s_1,s_2)$ [familiar from the theory of error-correcting codes (Peterson and Weldon, 1972)], defined as the number of digits (bits) which differ between the binary sequences s_1 and s_2. [Thus in Boolean form, $H(s_1,s_2)=\#_1(s_1\oplus s_2)$.] Particular configurations correspond to points in the space of all possible configurations. Under cellular automaton evolution, each initial configuration traces out a trajectory in time. If cellular automaton evolution is "stochastic," then the trajectories of nearby points (configurations) must diverge (exponentially) with time. Consider first the case of two initial configurations (say, S_1 and S_2) which differ by a change in the value at one site (and are thus separated by unit Hamming distance). After τ time steps of cellular automaton evolution, this initial difference may affect the values of at most 2τ sites (so that $H \leq 2\tau$). However, for simple cellular automaton rules, the difference remains localized to a few sites, and the total Hamming distance tends rapidly to a small constant value. The behavior for complex cellular automaton rules differs radically between additive rules (such as 90 and 150) and nonadditive ones. For additive rules, the difference obtained after τ time steps is given simply by the evolution of the initial difference (in this case a single nonzero site) for τ time steps. The Hamming distance at time step τ is thus given by the number of nonzero sites in the configuration obtained by evolution from a single site, and for rule 90 has the form $H_\tau=2^{\#_1(\tau)}$, as illustrated in Fig. 20(a). The average Hamming distance, smoothed over many time steps,

---

[9] An alternative specification would take each configuration to correspond to one of the 2^N vertices of an N-dimensional hypercube, labeled by coordinates corresponding to the values of the N sites. Points corresponding to configurations differing by values at a single site are then separated by a unit distance in N-dimensional space.
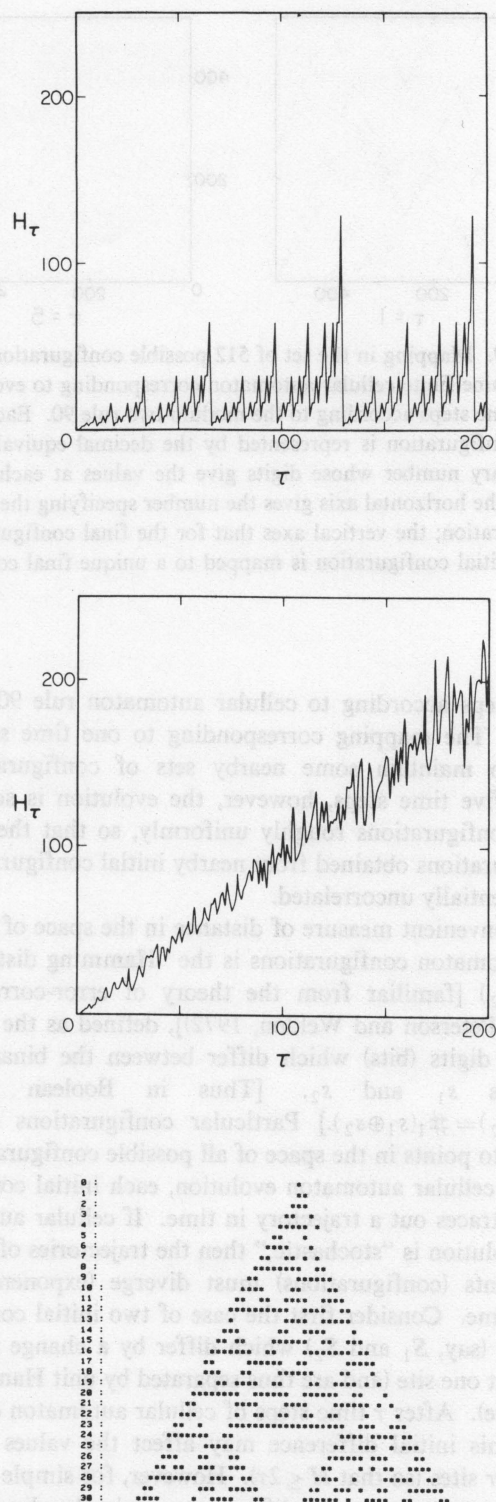
FIG. 19. ...

FIG. 20. Divergence in behavior of disordered configurations initially differing by a change in the value of a single site under cellular automaton evolution. The Hamming distance H between two configurations is defined as the number of bits (site values) which differ between the configurations. (a) shows the evolution of the Hamming distance between two configurations of the additive cellular automaton 90 (modulo-two rule); (b) shows the corresponding Hamming distance for the nonadditive cellular automaton 126; and (c) gives the actual difference (modulo two) between the configurations of cellular automaton 126 for the first few time steps. For nonadditive rules [case (b)], $H_\tau \sim \tau$, while for additive rules [case (a)], after time averaging, $H_\tau \sim \tau^{0.59}$.

behaves as $H_\tau = \tau^{\log_2 3 - 1} \simeq \tau^{0.59}$. For nonadditive rules, the difference between configurations obtained through cellular automaton evolution no longer depends only on the difference between the initial configurations. Figure 20(c) shows the difference between configurations obtained by evolution according to the nonadditive cellular automaton rule 126. The lack of symmetry in the pattern is a reflection of the dependence on the values of multiple initial sites. Figure 20(b) shows the Hamming distance corresponding to this difference. Apart from small fluctuations, it is seen to increase linearly with τ, tending at large τ to the form $H_\tau \simeq \tau$. This Hamming distance is the same as would be obtained by comparing sequences of 2τ sites in two disordered configurations with density 0.5. Thus a change in the value of a small number of initial sites is amplified by the evolution of a nonadditive cellular automaton, and leads to configurations with a linearly increasing number of essentially uncorrelated sites. (Changes in single sites may sometimes be eradicated after a single time step; this exceptional behavior occurs for cellular automaton rule 18, but is always absent if more than one adjacent site is reversed.) A bundle of initial trajectories therefore diverges with time into an exponentially increasing volume.

One may specify a statistical ensemble of states for a finite cellular automaton by giving the probability for each of the 2^N possible configurations. In a collection of many disordered states with density $\rho = \frac{1}{2}$, each possible cellular automaton configuration is asymptotically populated with equal probability. Such a collection of states will be termed an "equiprobable ensemble," and may be considered "completely disorganized." Cellular automaton evolution modifies the probabilities for states in an ensemble, thereby generating "organization." Figure 21 shows the probabilities for the 1024 possible configurations of a finite cellular automaton with $N = 10$ obtained after evolution for ten time steps according to rule 126 from an initial equiprobable ensemble. Figure 22 shows the evolution of these probabilities over ten time steps for several complex cellular automata. At each time step, dots are placed in positions corresponding to configurations occurring with nonzero probabilities. At $\tau = 0$, all configurations are taken to be equally probable. Cellular automaton evolution modifies the probabilities for different configurations, reducing the probabilities for some to zero, and leading to "gaps" in Fig. 22. In the initial ensemble, all configurations were assigned equal *a priori* probabilities. After evolution (or "processing") for a few time steps, an equilibrium ensemble is attained in which different configurations carry different probabilities, according to a definite distribution. Properties of the more probable configurations dominate statistical averages over the ensemble, giving rise to the distinctive average local features of equilibrium configurations described in Sec. III.

In the limit $N \to \infty$, a cellular automaton configuration may be specified by real number in the interval 0 to 1 whose binary decomposition consists of a sequence of digits corresponding to the values of the cellular automa-
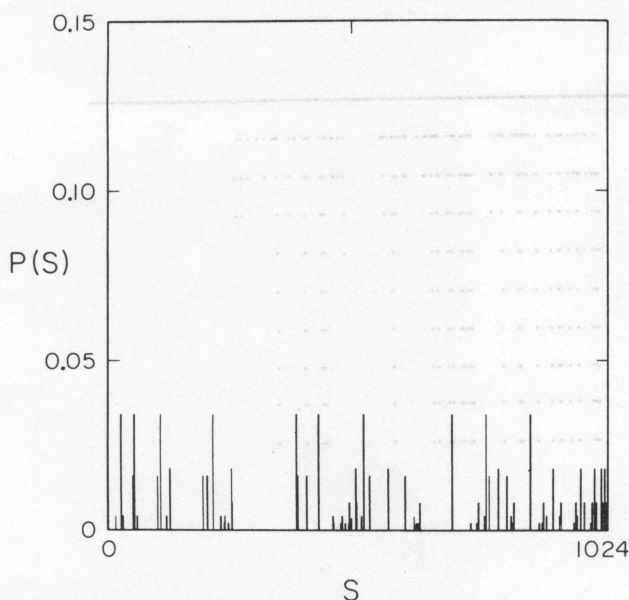
FIG. 21. Probabilities for each of the 1024 possible configurations in a finite (circular) cellular automaton with length $N = 10$ obtained by evolution according to rule 126 for ten time steps from an initial ensemble containing each possible configuration with equal probability. On the horizontal axis, each configuration S is labeled by a ten-digit binary integer (marked in decimal form) whose digits give the values of the corresponding sites. The null configuration (with value zero at all sites) is labeled by the integer 0, and occurs with the largest probability $\simeq 0.13$. The inequality of the probabilities for initially equiprobable configurations is a reflection for self-organization.

ton sites. Then the equilibrium ensemble of cellular automaton configurations analogous to those of Fig. 22 corresponds to a set of points on the real line. The unequal probabilities for appearance of 0 and 1 digits, together with higher-order correlations, implies that the points form a Cantor set (Farmer, 1982a, 1982b). The fractal dimensionality of the Cantor set is given by the negative of the entropy discussed below, associated with the ensemble of cellular automaton configurations (and hence real-number binary digit sequences) (Farmer, 1982a, 1982b). For rule 126 the fractal dimension of the Cantor set is then 0.5.

An important feature of the elementary cellular automata considered here and in Sec. III is their "local irreversibility." Cellular automaton rules may transform several different initial configurations into the same final configuration. A particular configuration thus has unique descendants, but does not necessarily have unique ancestors (predecessors). Hence the trajectories traced out by the time evolution of several cellular automaton configurations may coalesce, but may never split. A trivial example is provided by cellular automaton rule 0, under which all possible initial configurations evolve after one time step to the unique null configuration. In a reversible system, each state has a unique descendant and a unique ancestor, so that trajectories representing time evolution of different states may never intersect or meet. Thus in a reversible system, the total number of possible configura-

tions must remain constant with time (Liouville's theorem). However, in an irreversible system, the number of possible configurations may decrease with time. This effect is responsible for the "thinning" phenomenon visible in Fig. 22. The trajectories corresponding to the evolution of cellular automaton configurations are found to become concentrated in limited regions, and do not asymptotically fill the available volume densely and uniformly. This behavior makes self-organization possible, by allowing some configurations to occur with larger probabilities than others even in the large-time equilibrium limit.

One consequence of local irreversibility evident from Fig. 22 is that some cellular automaton configurations may appear as initial conditions but may never be reached as descendants of other configurations through cellular automaton time evolution.[10] Such configurations carry zero weight in the ensemble obtained by cellular automaton evolution. In the trivial case of cellular automaton rule 0, only the null state with all sites zero may be reached by time evolution; all other configurations are unreachable. Rule 4 generates only those configurations in which no two adjacent sites have the same value. The fraction of the 2^N possible configurations which satisfy this criterion tends to zero as N tends to infinity, so that in this limit, a vanishingly small fraction of the configurations are reached. Cellular automaton rule 204 is an identity transformation, and is unique among cellular automaton rules in allowing all configurations to be reached. (The rule is trivially reversible.) Assuming periodic boundary conditions, one finds that with N odd, the complex additive rule 90 generates only configurations in which an even number of sites have value one, and thus allows exactly half of the 2^N possible configurations to be reached. For even N, $\frac{1}{4}$ of the possible configurations may be reached. A finite fraction of all the configurations are thus reached in the limit $N \to \infty$. For the complex nonadditive rule 126, inspection of Fig. 8 shows that only configurations in which nonzero sites appear in pairs may be reached. Figure 23 shows the fraction of unreachable configuration for this cellular automaton rule as a function of N. The fraction tends steadily to one as $N \to \infty$. A complete characterization of the unreachable configurations for this case is given in Martin et al. (1983); these configurations are enumerated there, and their fraction is shown to behave as $1 - \lambda^N$ for large N, where $\lambda \simeq 0.88$ is determined as the root of a cubic equation. Similar behavior is found for other nonadditive rules.

Irreversible behavior in cellular automata may be analyzed by considering the behavior of their "entropy" S or "information content" $-S$. Entropy is defined as usual as the logarithm (here taken to base two) of the average

----

[10] The existence of unreachable or "garden-of-Eden" configurations in cellular automata is discussed in Moore (1962) and Aggarwal (1973), where criteria (equivalent to irreversibility) for their occurrence are given.

RULE : 00010010   (18)

RULE : 01011010   (90)
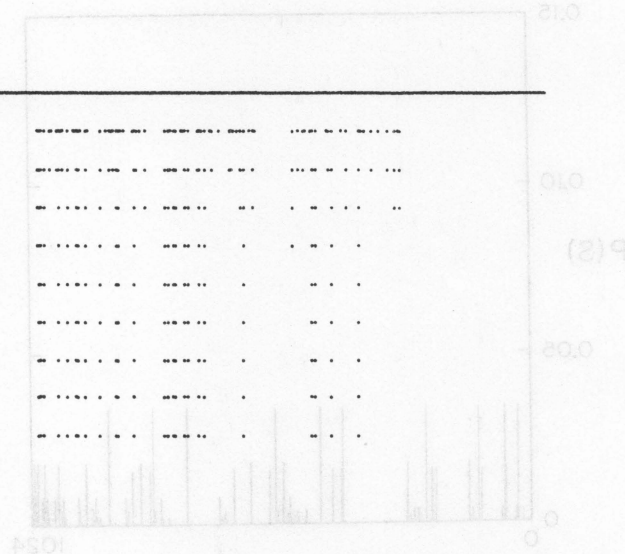
RULE : 01111110   (126)

FIG. 22. Time evolution of the probabilities for each of the 1024 possible configurations of several length 10 cellular automata starting from an initial ensemble containing all 1024 configurations with equal probabilities. The configurations are specified by binary integers whose digits form the sequence of values at the sites of the cellular automaton. The history of a particular configuration is given on successive lines in a vertical column: a dot appears at a particular time step if the configuration occurs with nonzero probability at that time step. In the initial ensemble, all configurations occur with equal nonzero probabilities, and dots appear in all positions. Cellular automaton evolution modifies the probabilities for the configurations, making some occur with zero probability, yielding gaps in which no dots appear. The probabilities obtained by evolution for ten time steps according to cellular automaton rule 126 were given in Fig. 21: dots appear in the tenth line of the rule 126 part of this figure at the positions corresponding to configurations with nonzero probabilities.
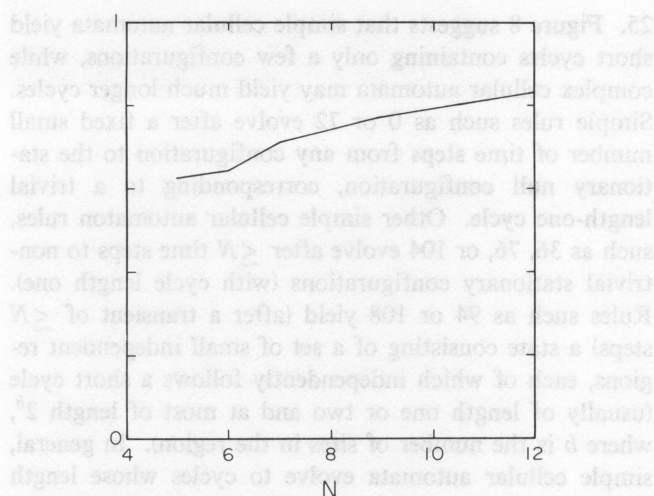
FIG. 23. Fraction of the 2^N possible configurations of a length N cellular automaton (with periodic boundary conditions) not reached by evolution from an arbitrary initial configuration according to cellular automaton rule 126. The existence of unreachable configurations is a consequence of the irreversibility of cellular automaton evolution. The fraction of such configurations is seen to increase steadily towards one as N increases.

number of possible states of a system, or

$$S = \sum_i p_i \log_2 p_i \tag{4.1}$$

where p_i is the probability for state i. The entropy may equivalently be considered as the average number of binary bits necessary to specify one state in an ensemble of possible states. The total entropy of a system is the sum of the entropies of statistically independent subsystems. Entropy is typically maximized when a system is completely disorganized, and the maximum number of subsystems act independently. The entropy of a cellular automaton takes on its maximal value of one bit per site
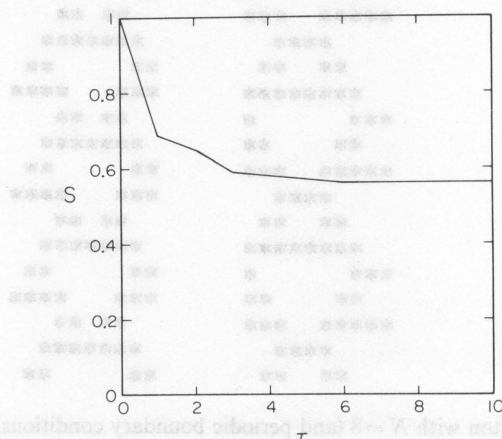
FIG. 24. Time evolution of average entropy per site for an ensemble of finite cellular automata with $N = 10$ evolving according to rule 126 from an initial equiprobable ensemble. The entropy gives the logarithm of the average number of possible configurations. Its decrease with time is a reflection of the local irreversibility of the cellular automaton.

for an equiprobable ensemble. For reversible systems, time evolution almost always leads to an increase in entropy. However, for irreversible systems, such as cellular automata, the entropy may decrease with time. Figure 24 shows the time dependence of the entropy for a finite cellular automaton with $N = 10$, evolving according to rule 126, starting from an initial equiprobable ensemble. The entropy is seen to decrease with time, eventually reaching a constant equilibrium value. The decrease is a direct signal of irreversibility.

The entropy for a finite cellular automaton given in Fig. 24 is obtained directly from Eq. (4.1) by evaluating the probabilities for each of the finite set of 2^N possible configurations. For infinite cellular automata, enumeration of all configurations is no longer possible. However, so long as values of sufficiently separated sites are statistically independent, the average entropy per site may nevertheless be estimated by a limiting procedure. Define a "block entropy" [or "Renyi entropy" (Renyi, 1970; Farmer, 1982a, 1982b)]

$$S_b = (1/b) \sum_i p_i^{(b)} \log p_i^{(b)},$$

where $p_i^{(b)}$ denotes the probability for a sequence i of b values in an infinite cellular automaton configuration. The limit $S_{b \to \infty}$ gives the average total entropy per site. This limit is approached rapidly for almost all cellular automaton configurations, reflecting the exponential decrease of correlations with distance discussed in Sec. III. [Similar results are obtained in estimating the entropy of printed English from single letter, digram, trigram and so on frequencies (Shannon, 1951). Typical results (for example, for the text of this paper) are $S_1 \simeq 4.70$, $S_2 \simeq 4.15$, $S_3 \simeq 3.57$, and $S_\infty \sim 2.3$.]

Irreversibility is not a necessary feature of cellular automata. In the case of the elementary cellular automata considered here, the irreversibility results from the assumption that a configuration S_n at a particular time step n depends only on its immediate predecessor so that its evolution may be represented schematically by $S_n = F[S_{n-1}]$. Except in the trivial case of the identity transformation (rule 204), F is not invertible. The cellular automata are discrete analogs of systems governed by partial differential equations of first order in time (such as the diffusion equation), and exhibit the same local irreversibility. One may construct reversible one-dimensional cellular automata (Fredkin, 1982; Margolus, 1982)[11] by allowing a particular configuration to depend on the previous two configurations, in analogy with reversible second-order differential equations such as the wave equation. The evolution of these cellular automata may be represented schematically by $S_n = F[S_{n-1}] \oplus S_{n-2}$. The

---

[11] Reversible cellular automata may be constructed in two (or more) dimensions by allowing arbitrary evolution along a line, but generating a sequence of copies ("history") in the orthogonal direction of the configurations on the line at each time step (Toffoli, 1977a, 1980).

invertibility of modulo-two addition allows S_{n-2} to be obtained uniquely from S_n and S_{n-1}, so that all pairs of successive configurations have unique descendants and unique ancestors. For infinite reversible cellular automata, the entropy (4.1) (evaluated for the appropriate successive pairs of configurations) almost always increases with time. Finite reversible cellular automata may exhibit globally irreversible behavior when dissipative boundary conditions are imposed. Such boundary conditions are obtained if sites beyond the boundary take on random values at each time step. If all sites beyond the boundary have a fixed or predictable value as a function of time, the system remains effectively reversible. With simple initial configurations, reversible cellular automata generate self-similar patterns analogous to those found for irreversible ones.[12] A striking difference is that reversible rules yield diamond-shaped structures symmetrical in time, rather than the asymmetrical triangle structures found with irreversible rules.

Since a finite cellular automaton has a total of only 2^N possible configurations, the sequence of configurations reached by evolution from any initial configuration must become periodic after at most 2^N time steps (the "Poincare recurrence time"). After an initial transient, the cellular automaton must enter a cycle in which a set of configurations is generated repeatedly, as illustrated in Fig. 25. Figure 8 suggests that simple cellular automata yield short cycles containing only a few configurations, while complex cellular automata may yield much longer cycles. Simple rules such as 0 or 72 evolve after a fixed small number of time steps from any configuration to the stationary null configuration, corresponding to a trivial length-one cycle. Other simple cellular automaton rules, such as 36, 76, or 104 evolve after $\lesssim N$ time steps to nontrivial stationary configurations (with cycle length one). Rules such as 94 or 108 yield (after a transient of $\lesssim N$ steps) a state consisting of a set of small independent regions, each of which independently follows a short cycle (usually of length one or two and at most of length 2^b, where b is the number of sites in the region). In general, simple cellular automata evolve to cycles whose length remains constant as N increases. On the other hand, complex cellular automata may yield cycles whose length increases without bound as N increases. Figure 26 shows the distribution in the number of time steps before evolution from each possible initial configuration according to the complex rule 126 leads to repetition of a configuration. Only a small fraction of the 2^N possible configurations is seen to be reached in evolution from a particular initial configuration. For example, in the case $N=8$, a maximum of eight distinct configurations (out of 256) are generated by evolution from any specific initial state.
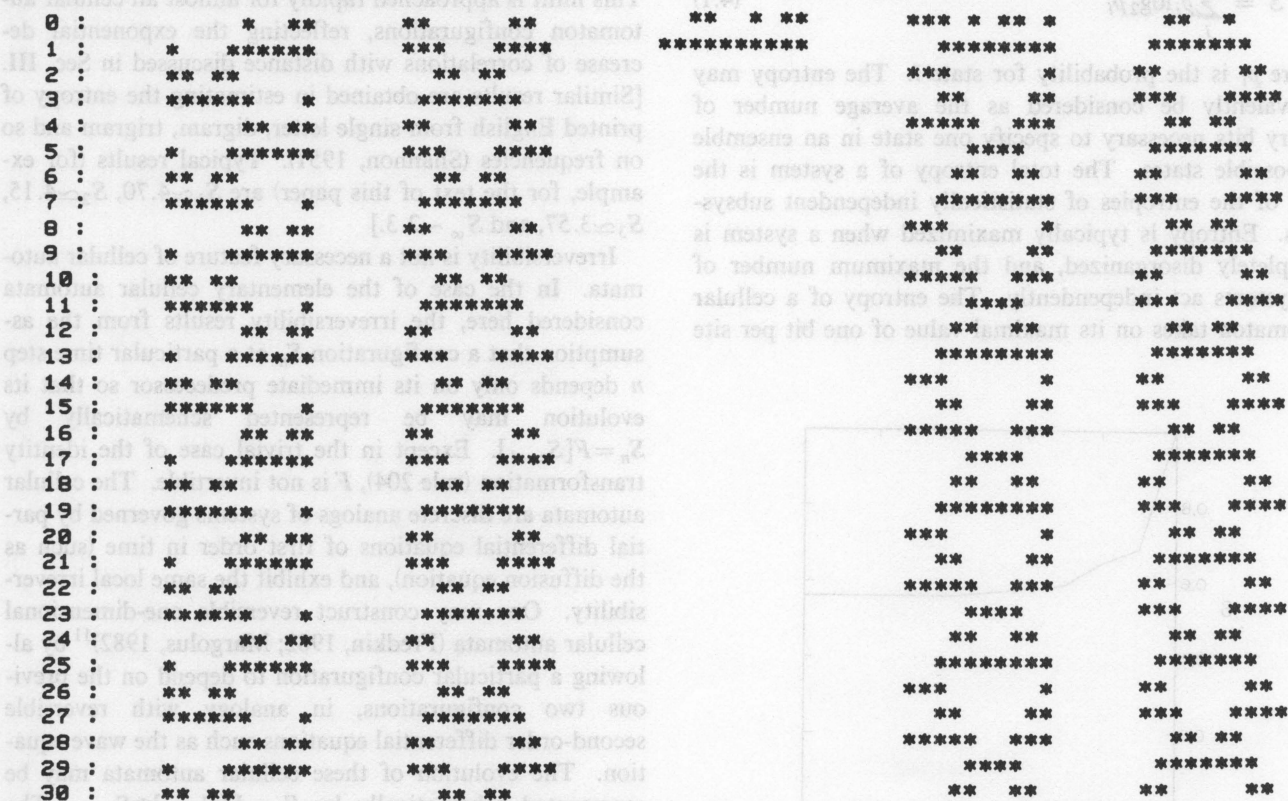

FIG. 25. Evolution of typical initial configurations in a finite cellular automaton with $N=8$ (and periodic boundary conditions) according to rule 126. Evolution from a particular initial state could generate up to $2^8=256$ distinct configurations before entering a cycle and returning to a configuration already visited. Much shorter cycles, however, are seen to occur in practice.

[12] For example, evolution from a pair of successive configurations containing zero and one nonzero sites according to the reversible analog of rule 150 yields a self-similar pattern with fractal dimension $\log_2[4/(\sqrt{17}-3)] \simeq 1.84$.
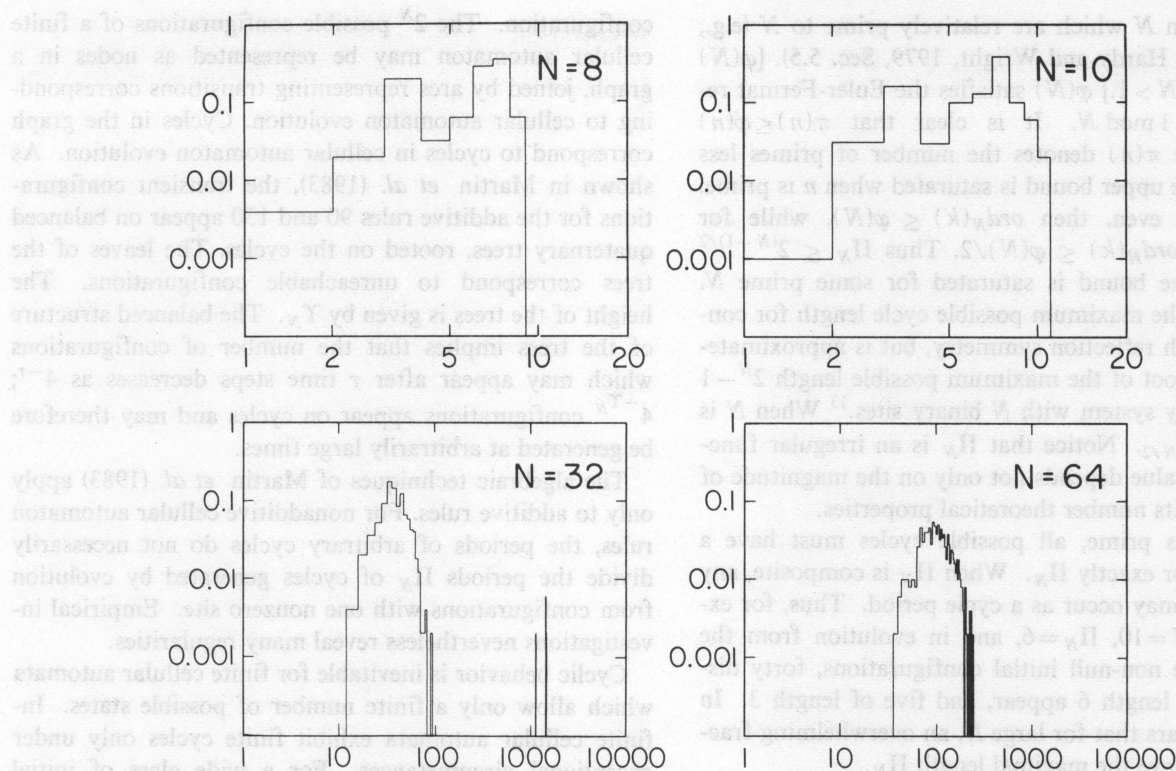
FIG. 26. Distribution in the number of time steps required for finite cellular automata of length N (with periodic boundary conditions) evolving according to rule 126 to reach a particular configuration for the second time, signaling the presence of a cycle. The cycle times found are much smaller than the value 2^N obtained if evolution from a particular initial configuration eventually visited all 2^N possible configurations. The results for $N=8$ and $N=10$ include all 256 and 1024 possible initial configurations; those for $N=32$ and $N=64$ are obtained by uniform Monte Carlo sampling from the space of possible initial configurations. In all cases, the number of configurations visited in transients before entering a cycle is very much smaller than the number of configurations in the cycle.

After a transient of at most two time steps, the cellular automaton enters a cycle, which repeats after at most six further time steps. Apart from the trivial one-cycle corresponding to the null configuration, six distinct cycles (containing nonintersecting sets of configurations) occur. Four have length six, and two have length two. A total of 29 distinct "final" configurations appear in these cycles. The number of configurations reached by evolution from a particular initial state increases with N as shown in Fig. 26. For $N=10$, the maximum is 38 states, while for $N=32$, it is at least 1547. Similar behavior is found for most other complex nonadditive rules.

Analytical results for transient and cycle lengths may be given for finite cellular automata (with periodic boundary conditions) evolving according to the additive rules 90 and 150 (Martin *et al.*, 1983). A complete and general derivation may be obtained using algebraic methods and is given in Martin *et al.* (1983). The additive superposition principle implies that the evolution of any initial configuration is a superposition of evolution from single nonzero sites (in each of the N cyclically equivalent possible positions). The period of any cycle must therefore divide the period Π_N obtained by evolution from a single nonzero site. Similarly, the length of any transient must divide the length Υ_N obtained with a single nonzero initial site. It is found that Π_N is identical for rules 90 and 150, but Υ_N in general differs. The first few values of Π_N for rules 90 and 150 (for $N=3$ through $N=30$) are 1,

1, 3, 2, 7, 1, 7, 6, 31, 4, 63, 14, 15, 1, 15, 14, 511, 12, 63, 62, 2047, 8, 1023, 126, 511, 28, 16383, and 30. Consider rule 90; derivations for rule 150 are similar. Whenever N is of the form 2^α, the cellular automaton ultimately evolves from any initial configuration to the null configuration, so that $\Pi_N=1$ in this case. When N is odd, it is found that the first configuration in the cycle always consists of two nonzero sites, separated by a single zero site. The nonzero sites may be taken at positions ± 1 modulo N. Equation (3.2) implies that configurations obtained by evolution for 2^j time steps again contain exactly two nonzero sites, at positions $\pm 2^j$ modulo N. A cycle occurs when $2^j \equiv \pm 1 \bmod N$. Π_N then divides Π_N^* given by $2^{sord_N(2)} - 1$ where $sord_N(k)$ is defined as the minimum j for which $2^j = \pm 1 \bmod N$, and $sord_N(k)=ord_N(k)/2$ or $sord_N(k)=ord_N(k)$. The multiplicative order function $ord_N(k)$ (e.g., MacWilliams and Sloane, 1977) is defined as the minimum j for which $2^j=1 \bmod N$. It is found in fact that $\Pi_N=\Pi_N^*$ for most N; the first exception occurs for $N=37$, in which case $\Pi_{37}=\Pi_{37}^*/3$. For $N=k^\alpha-1$, $ord_N(k)=\alpha$, so that when $N=2^\alpha-1$, $\Pi_N^*=N$. Similarly, when $N=k^\alpha+1$, $k^\alpha \equiv -1 \bmod N$ so that $k^{2\alpha} \equiv +1 \bmod N$ and $ord_N(k)=2\alpha$, yielding $\Pi_N^*=N-2$ for $N=2^\alpha+1$. In general, if $N=p_1^{\alpha_1} p_2^{\alpha_2} \cdots$, where the p_i are primes not equal to k, $ord_N(k)= lcm[ord_{p_1^{\alpha_1}}(k), ord_{p_2^{\alpha_2}}(k), \ldots]$. $ord_N(k)$ divides the Euler totient function $\varphi(N)$, defined as the number of in-

tegers less than N which are relatively prime to N (e.g., Apostol, 1976; Hardy and Wright, 1979, Sec. 5.5). [$\varphi(N)$ is even for all $N > 1$.] $\varphi(N)$ satisfies the Euler-Fermat relation $k^{\varphi(N)} \equiv 1 \bmod N$. It is clear that $\pi(n) \leq \varphi(n) \leq n - 1$, where $\pi(n)$ denotes the number of primes less than n, and the upper bound is saturated when n is prime. If $ord_N(k)$ is even, then $ord_N(k) \leq \varphi(N)$, while for $ord_N(k)$ odd, $ord_N(k) \leq \varphi(N)/2$. Thus $\Pi_N \leq 2^{(N-1)/2} - 1$, where the bound is saturated for some prime N. Such a Π_N is the maximum possible cycle length for configurations with reflection symmetry, but is approximately the square root of the maximum possible length $2^N - 1$ for an arbitrary system with N binary sites.[13] When N is even, $\Pi_N = 2\Pi_{N/2}$. Notice that Π_N is an irregular function of N: its value depends not only on the magnitude of N, but also on its number theoretical properties.

When Π_N is prime, all possible cycles must have a period of one or exactly Π_N. When Π_N is composite, any of its divisors may occur as a cycle period. Thus, for example, with $N = 10$, $\Pi_N = 6$, and in evolution from the $2^{10} - 1$ possible non-null initial configurations, forty distinct cycles of length 6 appear, and five of length 3. In general it appears that for large N, an overwhelming fraction of cycles have the maximal length Π_N.

As mentioned above, for the additive rules 90 and 150, the length of the transients before a cycle is entered in evolution from an arbitrary initial configuration must divide Υ_N, the length of transient with a single nonzero initial site. For rule 90, $\Upsilon_N = 1$ for N odd, and $\Upsilon_N = D_2(N)/2$ otherwise, where $D_2(n)$ is the largest 2^j which divides n. For rule 150, $\Upsilon_N = 0$ if N is not a multiple of three, $\Upsilon_N = 1$ if N is odd, and $\Upsilon_N = D_2(N)$ otherwise. Since, as discussed above, evolution from all 2^N possible initial configurations according to rule 90 visits 2^{N-1} configurations for odd N, the result $\Upsilon_N = 1$ implies that in this case, exactly half of the 2^N possible configurations appear on cycles.

Configurations in cellular automata may be divided into essentially three classes according to the circumstances under which they may be generated. One class discussed above consists of configurations which can appear only as initial states, but can never be generated in the course of cellular automaton evolution. A second class contains configurations which cannot arise except within the first, say τ, time steps. For $\tau = 2$, such configurations have "parents" but no "grandparents." The third class of configurations is those which appear in cycles, and may be visited repeatedly. Such configurations may be generated at any time step (for example, by choosing an initial configuration at the appropriate point in the cycle, and then allowing the necessary number of cycle steps to occur). The second class of configurations appears as transients leading to cycles. The cycles may be considered as attractors eventually attained in evolution from any initial

configuration. The 2^N possible configurations of a finite cellular automaton may be represented as nodes in a graph, joined by arcs representing transitions corresponding to cellular automaton evolution. Cycles in the graph correspond to cycles in cellular automaton evolution. As shown in Martin *et al.* (1983), the transient configurations for the additive rules 90 and 150 appear on balanced quaternary trees, rooted on the cycles. The leaves of the trees correspond to unreachable configurations. The height of the trees is given by Υ_N. The balanced structure of the trees implies that the number of configurations which may appear after τ time steps decreases as $4^{-\tau}$; $4^{-\Upsilon_N}$ configurations appear on cycles and may therefore be generated at arbitrarily large times.

The algebraic techniques of Martin *et al.* (1983) apply only to additive rules. For nonadditive cellular automaton rules, the periods of arbitrary cycles do not necessarily divide the periods Π_N of cycles generated by evolution from configurations with one nonzero site. Empirical investigations nevertheless reveal many regularities.

Cyclic behavior is inevitable for finite cellular automata which allow only a finite number of possible states. Infinite cellular automata exhibit finite cycles only under exceptional circumstances. For a wide class of initial states, simple cellular automaton rules can yield nontrivial cyclic behavior. Cycles occur in complex cellular automata only with exceptional initial conditions. Any initial configuration with a finite number of nonzero sites either evolves ultimately to the null state, or yields a pattern whose size increases progressively with time. Most infinite initial configurations do not lead to cyclic behavior. However, if the values of the initial sites form an infinite periodic sequence (c.f. Miller, 1970, 1980), with period k, then the evolution of the infinite cellular automaton will be identical to that of a finite cellular automaton with $k = N$, and cycles with length $\ll 2^k$ will be found.

The transformation of a finite cellular automaton configuration according to cellular automaton rules defines a mapping in the set of 2^N binary integers representing the cellular automaton configurations. An example of such a mapping was given in Fig. 19. Repeated applications of the mapping yield successive time steps in the evolution of the cellular automaton. One may compare the results with those obtained for a system which evolves by iteration of a random mapping among the 2^N integers (cf. Kauffman, 1969). Random mappings of K elements are obtained by choosing one of the K possible images independently for each integer and with equal probabilities. The mapping is permitted to take an element to itself. In this way, all K^K possible mappings are generated with equal probability. The probability of a particular element's having no preimage (predecessor) under a random mapping between K elements is $(K-1)^K / K^K = (1 - 1/K)^K$. In the limit $K \to \infty$ this implies that a fraction $1/e \simeq 0.37$ of the possible states are not reached in evolution by iteration of a random mapping. For complex nonadditive cellular automata, it appears that as $N \to \infty$, almost all configurations become unreachable, indicating that cellular automaton evolution

---

[13] The result is therefore to be contrasted with the behavior of linear feedback shift registers, analogous to cellular automata except for end effects, in which cycles (de Bruijn sequences) of period $2^N - 1$ may occur (e.g., Golomb, 1967; Berlekamp, 1968).

is "more irreversible" than iteration of a random mapping would imply. A system evolving according to a random mapping exhibits cycles analogous to those found in actual cellular automata. The probability of a length r cycle's occurring by iteration of a mapping between K elements is found to be

$$\sum_{i=r}^{K} \frac{(K-1)!}{(K-i)!K^i}$$

(Harris, 1960; Knuth, 1981, Sec. 3.1, Ex. 6, 11-16; Levy, 1982). Cycles of the maximum length K occur with finite probability. In the large K limit, the average cycle length becomes $\simeq\sqrt{\pi K/8}\simeq 0.63\sqrt{K}$, while the standard deviation of the cycle length distribution is $\simeq\sqrt{(2/3-\pi/8)K}\simeq 0.52\sqrt{K}$. The length of transients follows exactly the same distribution. The number of distinct cycles $\sim\sqrt{\pi/2}\log K$. If we take $K=256$ for comparison with an $N=8$ cellular automaton, this implies an average cycle length $\simeq 10$, an average transient length $\simeq 10$, $\simeq 94$ unreachable configurations, and $\simeq 7$ distinct cycles. Cellular automaton rule 126 yields in this case an average cycle length $\simeq 3.2$, an average transient length $\simeq 2.5$, 190 unreachable configurations, and 7 distinct cycles. Any agreement with results for random mappings appears to be largely fortuitous: even for large N cellular automata do not behave like random mappings.

This section has thus far considered cellular automata which evolve according to definite deterministic local rules. However, as discussed in Sec. III, one may introduce probabilistic elements or noise into cellular automata rules—for example, by reversing the value of a site at each time step with probability κ. Section III showed that the local properties of cellular automata change continuously as κ is increased from zero. Global properties may,

however, change discontinuously when a nonzero κ is introduced. An example of such behavior is shown in Fig. 27, which gives the fraction of configurations visited as a function of time for a cellular automaton evolving according to rule 126 with various values of κ, starting from a single typical initial configuration. When $\kappa=0$, only six distinct configurations are generated before the cellular automaton enters a cycle. When $\kappa\neq 0$, the cellular automaton ultimately visits every possible configuration (cf. Gach et al., 1978). For $\kappa\simeq 0.5$, one may approximate each configuration as being chosen from the 2^N possible configurations with equal probabilities: in this case, the average number of configurations visited after τ time steps is found to be $1 - ([1-2^{-N}]^{2^N})^{\tau/2^N} \simeq 1 - e^{-\tau/2^N}$.

Cellular automata may be viewed as simple idealizations of physical systems. They may also be interpreted as "computers" (von Neumann, 1966; Baer and Martinez, 1974; Burks, 1970; Aladyev, 1974, 1976; Toffoli, 1977b) and analyzed using methods from the formal theory of computation (Minsky, 1967; Arbib, 1969; Manna, 1974; Hopcroft and Ullman, 1979; Beckman, 1980). With this interpretation, the initial configuration of a cellular automaton represents a "program" and "initial data," processed by cellular automaton time evolution to give a configuration corresponding to the "output" or "result" of the "computation." The cellular automaton rules represent the basic mechanism of the computer; different programs may be "run" (or different "functions evaluated") by giving different initial or "input" configurations. This process is analogous to the "evolution" of the sequence of symbols on the tape of a Turing machine (Turing, 1936). However, instead of considering a single "head" which modifies one square of the tape at each time step, the cellular automaton evolution simultaneously affects all sites at each time step. As discussed in Sec. V, there exist "universal" cellular automata analogous to universal Turing machines, for which changes in the initial configuration alone allow any computable (or "recursive") function to be evaluated. A universal Turing machine may simulate any other Turing machine using an "interpreter program" which describes the machine to be simulated. Each "instruction" of the simulated machine is simulated by running the appropriate part of the interpreter program on the universal machine. Universal cellular automata may similarly simulate any other cellular automata. The interpreter consists of an encoding of the configurations for the cellular automaton to be simulated on the universal automaton. A crucial point is that so long as the encoding defined by the interpreter is sufficiently simple, the statistical characteristics of the evolution of configurations in the universal cellular automaton will be shared by the cellular automaton being simulated. This fact potentially forms the basis for universality in the statistical properties of complicated cellular automata.

The simplest encodings which allow one cellular automaton to represent or simulate others are pure substitution or "linear" ones, under which the value of a single site is represented by a definite sequence of site values.

FIG. 27. Fraction of configurations visited after τ time steps in a finite cellular automaton (with $N=7$) evolving from a single typical initial state according to rule 126 in the presence of noise which randomly reverses the values of sites at each time step with probability κ. When $\kappa=0$, the cellular automaton enters a cycle after visiting only six distinct configurations. When $\kappa\neq 0$, the cellular automaton eventually visits all 128 possible configurations.

(Such encodings are analogous to the correspondences between complex cellular automaton rules mentioned in Sec. III.) For example, a cellular automaton A evolving according to rule 22 may be used to simulate another cellular automaton B evolving according to rule 146. For every 0 in the initial configuration of B, a sequence 00 is taken in the initial configuration of A, and for every 1 in B, 01 is taken in A. Then after 2τ time steps, the configuration of A under this encoding is identical to that obtained by evolution of B for τ time steps. If cellular automaton B instead evolved according to rule 182, 01 (or 10) in A would correspond to 0 in B, and 00 to 1. The simplicity of the interpreter necessary to represent rules 146 and 182 under rule 22 is presumably responsible for the similarities in their statistical behavior found in Sec. III. Figure 28 gives a network which describes the simulation capabilities of the complex elementary cellular automaton rules using length two linear encodings and with the simulated rule running at half the speed of the simulator. Many of these complex rules may also simulate simple rules under such an encoding. Simulations possible with longer linear encodings appear to be described by indirection through the network. Not all complex cellular automaton rules are thus related by linear encodings of any length.

As discussed in Sec. V, the elementary cellular automata considered here and in Secs. II and III are not of sufficient complexity to be capable of universal computation. However, some of the more complicated cellular automata described in Sec. V are "universal," and may therefore in principle represent any other cellular automata. The necessary encoding must be of finite length, but may be very long. The shorter or simpler the encoding, the closer will be the statistical properties of the simulating and simulated cellular automata.

V. EXTENSIONS

The results of Secs. II—IV have for the most part been restricted to elementary cellular automata consisting of a sequence of sites in one dimension with each site taking on two possible values, and evolving at each time step according to the values of its two nearest neighbors. This
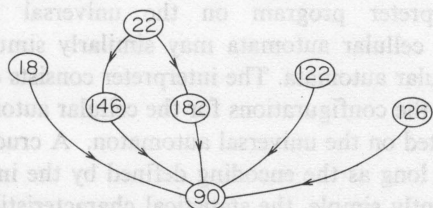
FIG. 28. Network describing simulation capabilities of complex elementary cellular automata with length two pure substitution or linear encodings. Cellular automata evolving according to the destination rule are simulated by giving an encoded initial configuration in a cellular automaton evolving according to the source rule. Representability of one cellular automaton by another under a simple encoding implies similar statistical properties for the two cellular automata, and forms potentially the basis for universality in statistical properties of cellular automata.

section gives a brief discussion of the behavior of more complicated cellular automata. Fuller development will be given in future publications.

We consider first cellular automata in which the number of possible values k at each site is increased from two, but whose sites are still taken to lie on a line in one dimension. The evolution of each site at each time step is for now assumed to depend on its own value and on the values of its two nearest neighbors. In this case, the total number of possible sets of local rules is $k^{(k^3)}$. Imposition of the reflection symmetry and quiescence "legality conditions" discussed in Sec. II introduces $\frac{1}{2}k^2(k-1)+1$ constraints, yielding $k^{[k^2(1+k)-1]/2}$ "legal" sets of rules. For $k=2$, this implies $2^5=32$ legal rules, as considered in Sec. II. The number of possible legal rules increases rapidly with k. For $k=3$, there are $3^{17}=129\,140\,163\simeq1.3\times10^8$ rules, for $k=4$, $\simeq3\times10^{24}$, and for $k=10$, 10^{549}.

As a very simple example of cellular automata with $k>2$, consider the family of "modulo-k" rules in which at each time step, the value of a site is taken to be the sum modulo k of the values of its two neighbors on the previous time step. This is a generalization of the modulo-two rule (90) discussed on several occasions in Secs. II—IV. Figure 29 shows the evolution of initial states containing a single site with value one according to several modulo-k rules. In all cases, the pattern of nonzero sites is seen to tend to a self-similar fractal figure in the large time limit. The pattern in general depends on the value of the nonzero initial site, but in all cases yields an asymptotically self-similar figure. When k is prime, independent of the value of the initial nonzero site, a very regular pattern is generated, in which the density $T(n)$ of "triangle structures" is found to satisfy a one-term recurrence relation yielding a fractal dimension

$$D_k = \log_k \sum_{i=1}^{k} i = 1 + \log_k \left[\frac{k+1}{2} \right],$$

so that $D_3=1+\log_3 2\simeq1.631$, $D_5\simeq1.683$, and so on. When k is a composite number, the pattern generated depends on the value s of the initial nonzero site. If the greatest common divisor (s,k) of k and s is greater than one (so that s and k share nontrivial prime factors), then the pattern is identical to that obtained by evolution from an initial site with value one according to a modulo-$k/(s,k)$ rule. In general, the density of triangles satisfies a multiple-term recurrence relation. In all cases, the fractal dimension for large k behaves as $D\sim2-1/\log_2 k$ [assuming $(s,k)\ll k$]. When $k\to\infty$, the values of sites become ordinary integers, all with nonzero values by virtue of the nonvanishing values of binomial coefficients, yielding a figure of dimension two.

All modulo-k rules obey the additive superposition principle discussed for the modulo two in Secs. II and III. The number of sites with value r after evolution for τ steps from an initial state containing a single site with value one is found [on analogy to Eq. (3.2)] to be $N_\tau^{(r)}=2^{\#_r^{[k]}(\tau)}$, where the function $\#_r^{[k]}(\tau)$ gives the number of occurrences of the digit r in the base-k decomposi-
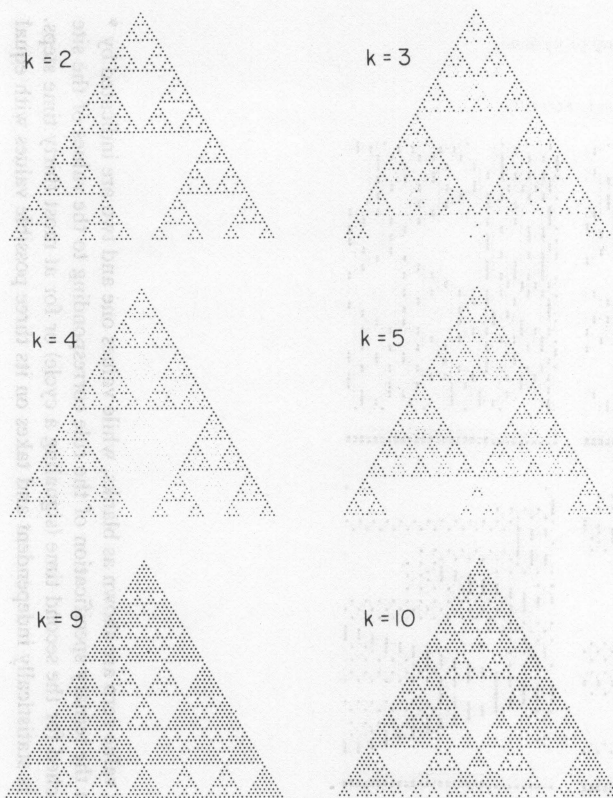
FIG. 29. Patterns generated by evolution of one-dimensional cellular automata with k states per site according to a modulo-k rule, starting from an initial configuration containing a single nonzero site with value one. At each time step, the value of a site is the sum of the values of its two nearest neighbors at the previous time step. Configurations obtained at successive time steps are shown on successive lines. Sites with value zero are indicated as blanks; $*$, $+$, and $-$ represent, respectively, values one, two, and three, and in the lower two patterns, ρ represents any nonzero value. In the large time limit, all the patterns tend to a self-similar form, with definite fractal dimensions.

tion of the integer τ and generalizes the function $\#_r(\tau)$ introduced in Sec. III.

Figure 30 shows typical examples of the behavior of some cellular automata with $k = 3$. Considerable diversity is evident. However, with simple initial states, self-similar patterns are obtained at asymptotically large times, just as in the $k = 2$ case of Sec. III. (Notice that the length and time scales before self-similarity becomes evident are typically longer than those found for $k = 2$: in the limit $k \to \infty$ where each site takes on an arbitrary integer value, self-similarity may not be apparent at any finite time.) Evolution of disordered initial states also again appears to generate nontrivial structure, though several novel phenomena are present. First, alternation of value-one and value-two sites on successive time steps can lead to "half-speed propagation" as in rule

00000000000000010020001010020 .

Second, rules such as

00000000000000010110020010010

lead to a set of finite regions containing only sites with values zero and one, separated by "impermeable mem-

branes" of value-two sites. The evolution within each region is independent, with the membranes enforcing boundary conditions, and leading to cycles after a finite number of time steps. Third, even for legal rules such as

00000012102200222100210201000

and

21100012212101220011202120000 ,

illustrated in Fig. 30, there exist patterns which display a uniform shifting motion. For example, with rule

21100012212101220011202120000

an isolated 12 shifts to the right by one site every time step, while an isolated 21 shifts to the left; when 21 and 12 meet, they cross without interference. Uniform shifting motion is impossible with legal rules when $k = 2$, since sequences of zero and one sites cannot define suitable directions (evolution of 1101 and 1011 always yield a pattern spreading in both directions).

An important feature of some cellular automata with more than two states per site is the possibility for the formation of a membrane which "protects" sites within it from the effects of noise outside. In this way, there may exist seeds from which very regular patterns may grow, shielded by membranes from external noise typical in a disordered configuration. Examples of such behavior are to be found in Fig. 30. Only when two protective membranes meet is the structure they enclose potentially destroyed. The size of the region affected by a particular seed may grow linearly with time. Even if seeds occur with very low probability, any sufficiently long disordered configuration will contain at least one, and the large time behavior of the cellular automaton will be radically affected by its presence.

In addition to increasing the number of states per site, the cellular automata discussed above may be generalized by increasing the number of sites whose values affect the evolution of a particular site at each time step. For example, one may take the neighborhood of each site to contain the site itself, its nearest neighbors, and its next-nearest neighbors. With two states per site, the number of possible sets of legal local rules for such cellular automata is $2^{26} \simeq 7 \times 10^7$ (for $k = 3$, this number increases to $3^{174} \simeq 10^{83}$). Figure 31 shows patterns generated by these cellular automata for two typical sets of local rules. With simple initial states, self-similar patterns are obtained at large times. With disordered initial states, less structure is apparent than in the three-site neighborhood cellular automata discussed above. The patterns obtained with such cellular automata are again qualitatively similar to those shown in Sec. II.

The cellular automata discussed so far have all involved a line of sites in one dimension. One may also consider cellular automata in which the sites lie on a regular square or (hyper)cubic lattice in two or more space dimensions. As usual, the value of each site is determined by the values of a neighborhood of sites at the previous time step. In the simplest case, the neighborhood includes a site
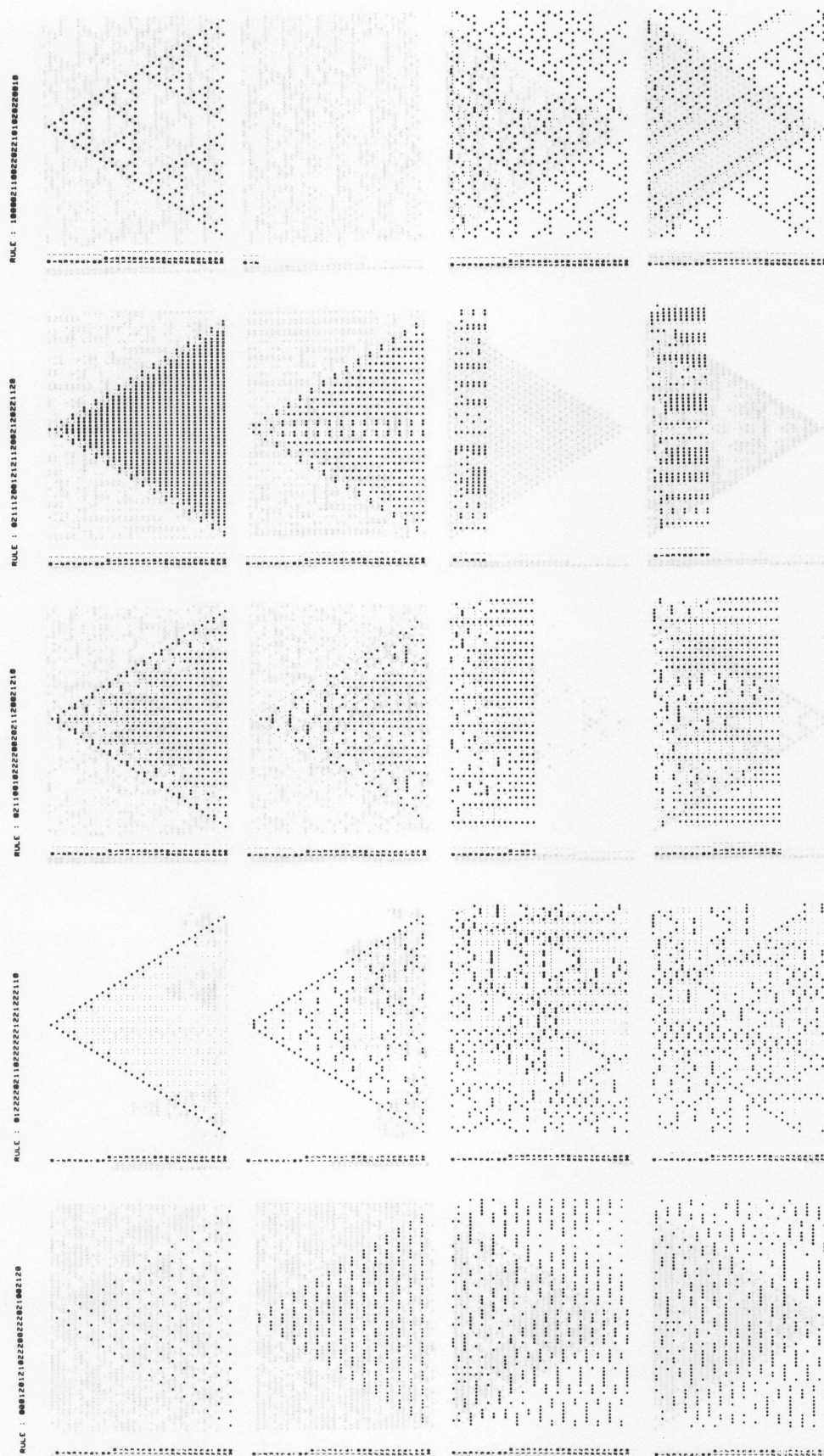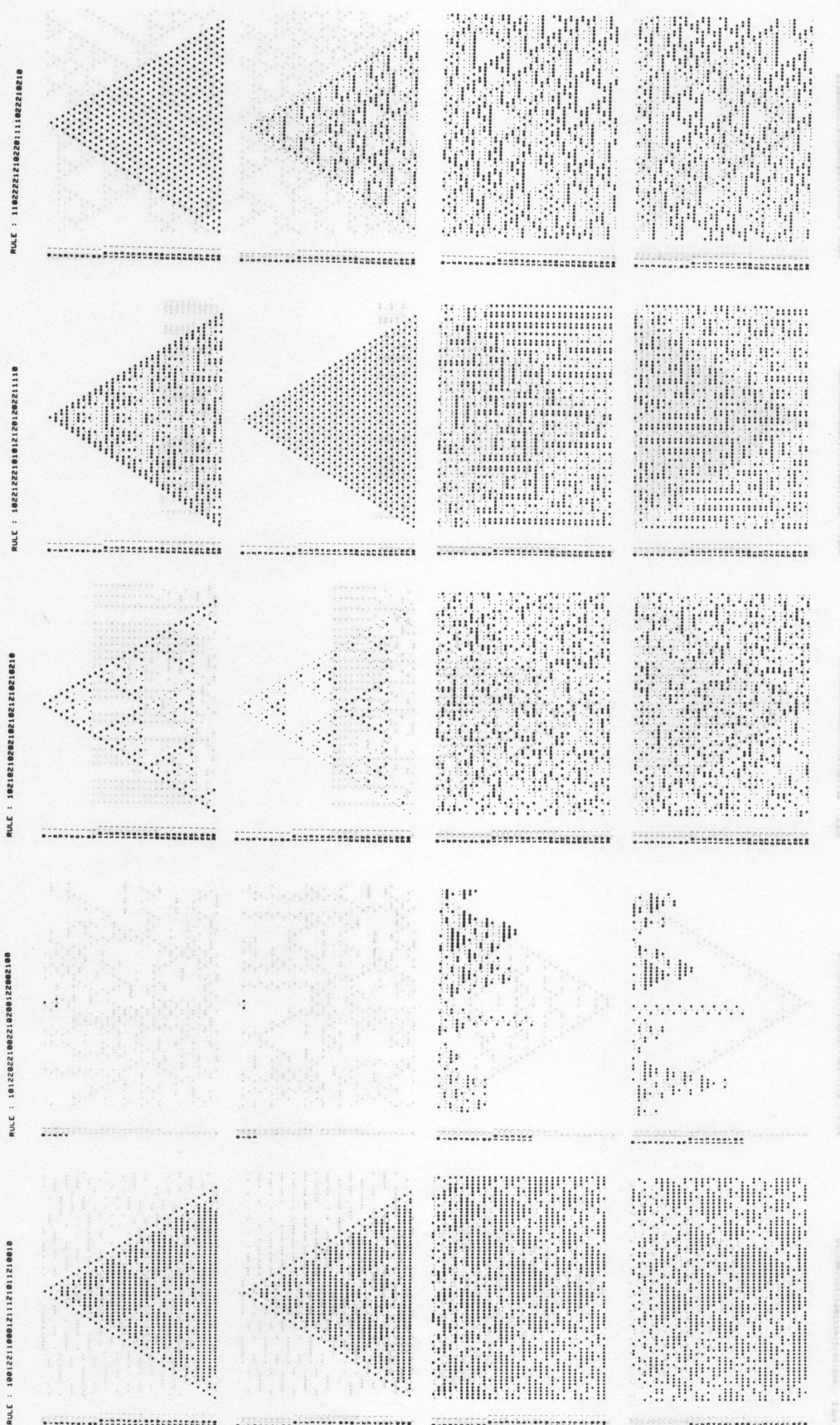
RULE : 0010000111021022122111020120

RULE : 0000000121022002210021020100

RULE : 0000000000000200100202022010

RULE : 0000000000000100200101000020

RULE : 0000000000000101100201000010

FIG. 30. Examples of the evolution of several typical cellular automata with three states per site. Sites with value zero are shown as blanks, while values one and two are indicated by *
and ·, respectively. The value of a site at each time step is determined in analogy with Fig. 1 by the digit in the ternary specification of the rule corresponding to the values of the site
and its two nearest neighbors at the previous time step. The evolution is shown until a configuration is reached for the second time (signaling a cycle) or for at most thirty time steps.
The initial configurations in the lower two rows are typical of disordered configurations in which each site is statistically independent and takes on its three possible values with equal
probabilities.

FIG. 30. (*Continued.*)

FIG. 30. (Continued.)

FIG. 30. (*Continued.*)

RULE : 10110101101101011101011101011101011011010
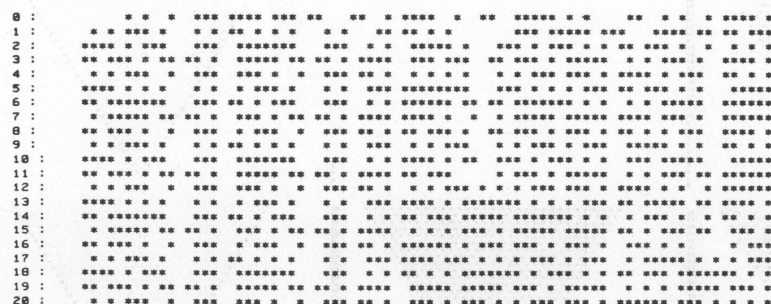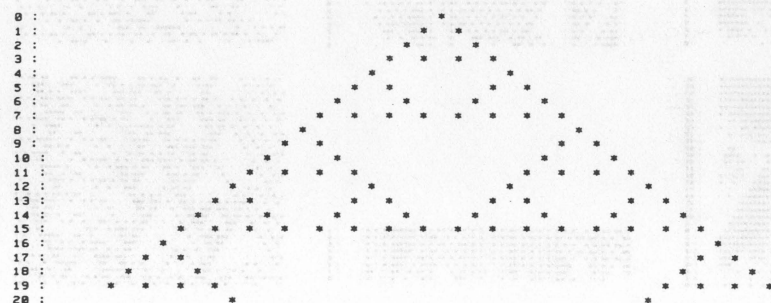
RULE : 0100101001001010010110110101010100
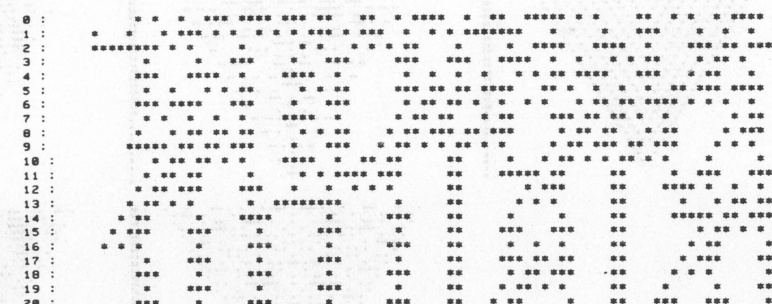
FIG. 31. Evolution of two typical one-dimensional cellular automata with two states per site in which the value of a site at a particular time step is determined by the preceding values of a neighborhood of five sites containing the site, its nearest neighbors, and its next-nearest neighbors. The initial configurations in the lower row are typical of disordered configurations, in which each site has value one with probability $\frac{1}{2}$.

and its nearest neighbors. However, in $d > 1$ dimensions two possible identifications of nearest neighbors can be made. First, sites may be considered neighbors if one of their coordinates differ by one unit, and all others are equal, so that the sites are "orthogonally" adjacent. In this case, a "type-I" cellular automaton neighborhood containing $2d + 1$ sites is obtained. Second, sites may be considered neighbors if none of their coordinates differ by more than one unit, so that the sites are "orthogonally" or "diagonally" adjacent. This case yields a "type-II" cellular automaton neighborhood containing 3^d sites. When $d = 1$, type-I and -II neighborhoods are identical and each contains three sites. For $d = 2$, the type-I neighborhood contains five sites, while the type-II neighborhood contains nine sites.[14] Cellular automaton rules may be considered legal if they satisfy the quiescence condition and are invariant under the rotation and reflection symmetries of the lattice. For $d = 2$, the number of possible legal type-I rules with k states per site is found to be $k^{(k^5 + k^3 + 2k^2 - 4)/4}$, yielding $2^{11} = 2048$ rules for $k = 2$ and $3^{71} \simeq 8 \times 10^{33}$ for $k = 3$. The number of type-II rules with $k = 2$ in two dimensions is found to be $2^{59} \simeq 6 \times 10^{17}$ (or 2^{71} if reflection symmetries are not imposed).

Figure 32 shows the evolution of an initial configuration containing a single nonzero site according to two-dimensional (type-I) modulo-two rules. In case (a) the value of a site is taken to be the sum modulo two of the values of its four neighbors on the previous time step, in analogy with one-dimensional cellular automaton rule 90. In case (b), the previous value of the site itself included in the sum (and the complement is taken), in analogy with rule 150. The sequence of patterns obtained at successive time steps may be "stacked" to form pyramidal structures in three-dimensional space. These structures become self-similar at large times: in case (a) they exhibit a fractal dimension $\log_2 5 \simeq 2.32$, and in case (b) a dimension $1 + \log_2(1 + \sqrt{3}) \simeq 2.45$. The patterns found on vertical slices containing the original nonzero site through the pyramids (along one of the two lattice directions) are the same as those generated by the one-dimensional modulo-two rules discussed in Secs. II and III. The patterns obtained at each time step in Fig. 31 are almost always self-similar in the large time limit. For case (a), the number of sites with value one generated after τ time steps in Fig. 31 is found to be $4^{\#_1(\tau)}$, where $\#_1(\tau)$ gives the number of occurrences of the digit one in the binary decomposition of the integer τ, as discussed in Sec. III (cf. Butler and Ntafos, 1977). The type-I modulo-two rules may be generalized to d-dimensional cellular automata. In case (a) the patterns obtained by evolution from a single nonzero initial site have fractal dimension $\log_2(2d + 1)$ and give $(2d)^{\#_1(\tau)}$ nonzero sites at time step τ. In case (b), the asymptotic fractal dimension is found to be $\log_2[d(\sqrt{1 + 4/d} + 1)]$. Once again, simple initial states always yield self-similar structures in the large time limit.

A particular type-II two-dimensional cellular automaton whose evolution has been studied extensively is the game of "Life" (Conway, 1970; Gardner, 1971, 1972; Wainwright, 1971–1973; Wainwright, 1974; Buckingham, 1978; Berlekamp et al., 1982, Chap. 25; R. W. Gosper, private communications). The local rules take a site to "die" (attain value zero) unless two or three of its neighbors are "alive" (have value one). If two neighbors are alive, the value of the site is left unchanged; if three are alive, the site always takes on the value one. Many configurations exhibiting particular properties have been found. The simplest isolated configurations invariant under time evolution are the "square" (or "block") consisting of four adjacent live sites, and the "hexagon" (or "beehive") containing six live sites. "Oscillator" configurations which cycle through a sequence of states are also known. The simplest is the "blinker" consisting of a line of three live sites, which cycles with a period of two time steps. Oscillators with periods 3, 5, and 7 are also known; other periods may be obtained by composition. So long as they are separated by four or more unfilled sites, many of these structures may exist without interference in the configurations of a cellular automaton, and their effects are localized. There also exist configurations which "move" uniformly across the lattice, executing a cycle of a few internal states. The simplest example is the "glider" which contains five live sites and undergoes a cycle of length two. The number of filled sites in all the configurations mentioned so far is bounded as a function of time. However, "glider gun" configurations have been found which generate infinite streams of gliders, yielding a continually increasing number of live sites. The simplest known glider gun configuration evolves from a configuration containing 26 live cells. Monte Carlo simulation suggests that a disordered state of N^2 cells usually evolves to a steady state within about N^2 time steps (and typically an order of magnitude quicker); very few of the 2^{N^2} possible configurations are visited. Complicated structures such as glider guns are very rarely produced. Rough empirical investigation suggests that the density of structures containing L live sites generated from a disordered initial state (cf. Buckingham, 1981) decreases like e^{-L_-}/L, where L_- is the size of the minimal distinct configuration which evolves to the required structure in one time step. Just as for the one-dimensional cellular automata discussed in Sec. IV, the irreversibility of "Life" leads to configurations which cannot be reached by evolution from any other configurations, and can appear only as initial states. However, the simplest known "unreachable" configuration contains around 300 sites (Wainwright, 1971–1973; Hardouin-Duparc, 1974; Berlekamp et al., 1982, Chap. 25).

The game of "Life" is an example of a special class of "totalistic" cellular automata, in which the value of a site depends only on the sum of the values of its neighbors at the previous time step, and not on their individual values. Such cellular automata may arise as models of systems involving additive local quantities, such as chemical concentrations. In one dimension with $k = 2$ (and three sites

---

[14] In the case $d = 2$, neighborhoods of types I and II are known as von Neumann and Moore neighborhoods, respectively.
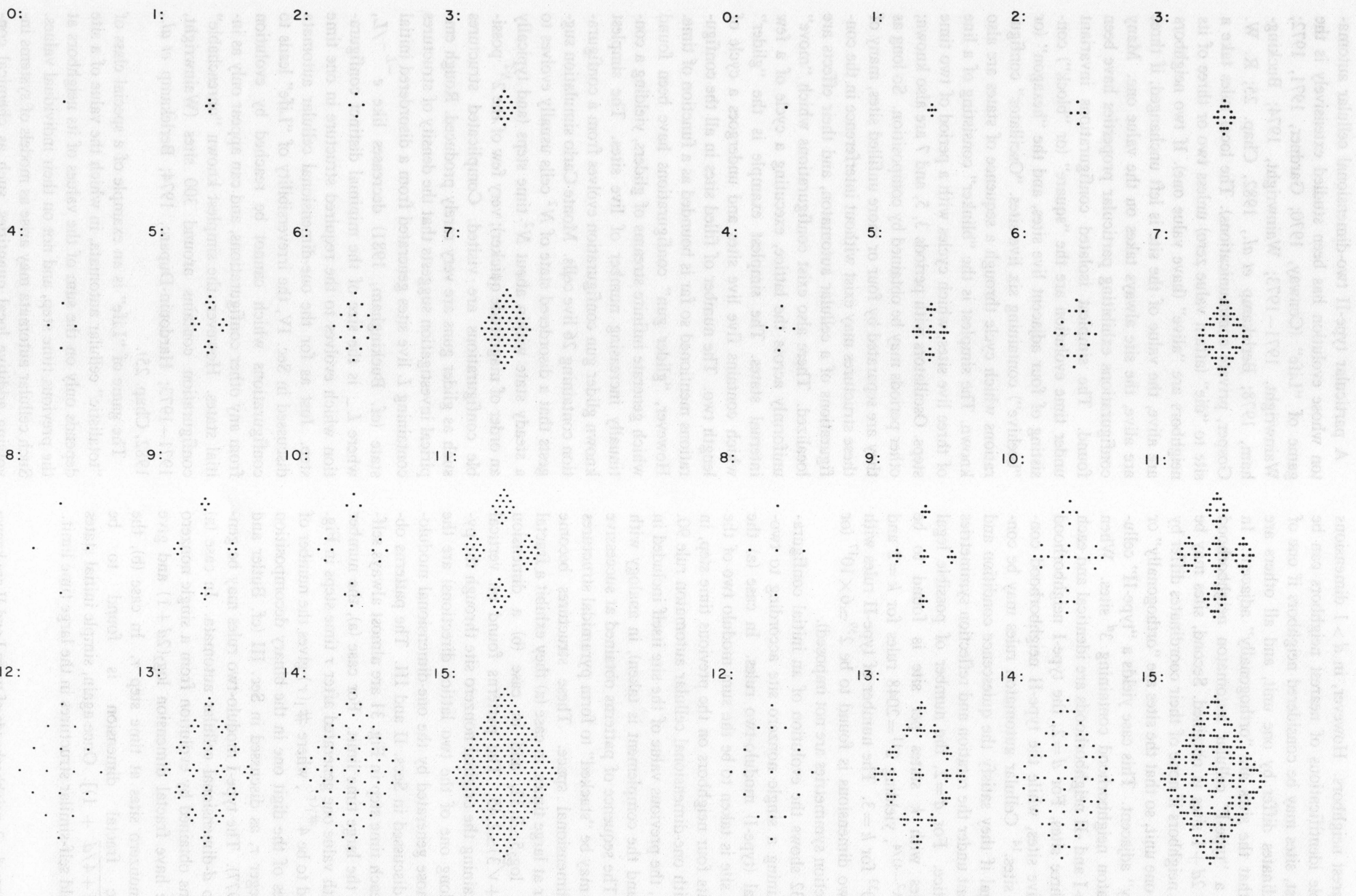
FIG. 32. Evolution of an initial state containing a single nonzero site in a two-dimensional cellular automaton satisfying type-I modulo-two rules. In case (a) the value of each site is taken to be the sum modulo two of the values of its four (orthogonally adjacent) neighbors at the previous time step, while in case (b) the previous value of the site itself is included in the sum, and the complement is taken. Case (a) is the two-dimensional analog of a one-dimensional cellular automaton evolving according to local rule 90, and case (b) of one evolving according to rule 150. The pyramidal structure obtained in each case by stacking the patterns generated at successive time steps is self-similar in the large time limit.

in each neighborhood) all cellular automaton rules are totalistic. In general, the number of totalistic (legal) sets of rules for cellular automata with v neighbors for each site is $k^{(k-1)(vk+1)}$. In one dimension with $k=3$, $\simeq 5 \times 10^6$ of the $\simeq 10^8$ possible rules are therefore totalistic. Only 243 of the totalistic rules are also periperal in the sense defined in Sec. II. With $k=2$ in two dimensions, 2^9 of the 2^{11} possible rules in a type-I neighborhood are totalistic (and 32 are also peripheral), and 2^{17} of the 2^{59} in a type-II neighborhood.

A potentially important feature of cellular automata is the capability for "self-reproduction" through which the evolution of a configuration yields several separated identical copies of the configuration. Figure 33 illustrates a very simple form of self-reproduction with the elementary one-dimensional modulo-two rule (see Waksman, 1969; Amoroso and Cooper, 1971; Fredkin, 1981). With a single nonzero site in the initial state, a configuration containing exactly two nonzero sites is obtained after 2^j time steps[15] as indicated by Eq. (3.2). The additive superposition property of the modulo-two rule implies that results for more complicated initial states are obtained by superposition of those for single-site initial states. Thus after $\tau = 2^j$ time steps, for sufficiently large j, the cellular automaton generates two exact copies of any initial sequence of site values. After a further 2^{j-1} time steps, four copies are obtained. However, after another 2^{j-1} time steps, the innermost pair of these copies meet again, and annihilate, leaving only two copies when $\tau = 2^{j+1}$. Purely geometrical "overcrowding" thus prevents exponential multiplication of copies by self-reproduction in this case. An exactly analogous phenomenon occurs with the two-dimensional modulo-two rule illustrated in Fig. 32, and its higher-dimensional analogs. In general, the number of sites in a d-dimensional cellular automaton configuration grows with time at most as fast as $(2\tau)^d$, which is asymptotically slower than the number $> (2d)^{\alpha\tau}$ required for an exponentially increasing number of copies to be generated. Exponential self-reproduction can thus occur only if the copies generated are not precisely identical, but exhibit variability, and for example execute a random walk motion in response to external noise or contain a "counter" which causes later generations to "live" longer before reproducing.

Section IV mentioned the view of cellular automata as computers. An important class of computers is those with the property of "computational universality," for which changes in input alone allow any "computable function" to be evaluated, without any change in internal construction. Universal computers can simulate the operation of any other computer if their input is suitably encoded. Many Turing machines have been shown to be computationally universal. The simplest has seven internal states, and allows four possible "symbols" in each square of its tape. One method for demonstrating computational universality of cellular automata shows correspondence with a universal Turing machine. The head of the Turing machine is typically represented by a phononlike structure which propagates along the cellular automaton. It may be shown (Smith, 1971) that an eighteen-state one-dimensional cellular automaton with a three-site neighborhood can simulate the seven-state four-symbol Turing machine in this way, and is therefore computationally universal. Simpler computationally universal cellular automata must be found by other methods. The most straightforward method is to show correspondence with a standard digital computer or electronic circuit by identifying cellular automaton structures which act like "wires," carrying signals without dissipation and crossing without interference, and structures representing NAND gates at intersections between wires. "Memories" which maintain the same state for all time are also required. In the Life-game cellular automaton discussed above, streams of gliders generated by glider guns may be used as wires, with bits in the signal represented by the presence or absence of gliders. At the points where "glider streams" meet, other structures determine whether the corresponding wires cross or interact through a "NAND gate." The Life-game cellular automaton is thus computationally universal. "Circuits" such as binary adders (Buckingham, 1978) may be constructed from Life configurations. It appears that such circuits run at a speed slower than the digital computers to which they correspond only by a constant multiplicative factor. The "Life game" is a type-II two-dimensional cellular automaton with two states per site. A computationally universal type-I two-dimensional cellular automaton has been constructed with three states per site (Banks, 1971); only two states are required if the initial configuration is permitted to contain an infinite "background" of nonzero sites (Toffoli, 1977a). In one dimension, with a neighborhood of three sites, there are some preliminary indications that a universal cellular automaton may be constructed with five states per site. The details and implications of this cellular automaton will be described in a future publication.

VI. DISCUSSION

This paper represents a first step in the investigation of cellular automata as mathematical models for self-organizing statistical systems. The bulk of the paper consisted in a detailed analysis of elementary cellular automata involving a sequence of sites on a line, with a binary variable at each site evolving in discrete time steps according to the values of its nearest neighbors. Despite the simplicity of their construction, these systems were found to exhibit very complicated behavior.

The 32 possible (legal) elementary cellular automata were found to fall into two broad classes. The first class consisted of simple cellular automata whose time evolu-

---

[15] An analogous result holds for all modulo-k rules with k prime by virtue of the relation $\binom{k^j}{i} \bmod k = 0$, $0 < i < k^j$ valid for all primes k. The relation is a special case of the general result (Knuth, 1973, Sec. 1.2.6, Ex. 10)

$$\begin{bmatrix} j \\ i \end{bmatrix} = \begin{bmatrix} \lfloor j/k \rfloor \\ \lfloor i/k \rfloor \end{bmatrix} \begin{bmatrix} j \bmod k \\ i \bmod k \end{bmatrix} \bmod k \; .$$

```
 0 :
 1 :
 2 :
 3 :
 4 :
 5 :
 6 :
 7 :
 8 :
 9 :
10 :
11 :
12 :
13 :
14 :
15 :
16 :
17 :
18 :
19 :
20 :
21 :
22 :
23 :
24 :
25 :
26 :
27 :
28 :
29 :
30 :
31 :
32 :
33 :
34 :
35 :
36 :
37 :
38 :
39 :
40 :
41 :
42 :
43 :
44 :
45 :
46 :
47 :
48 :
49 :
50 :
```
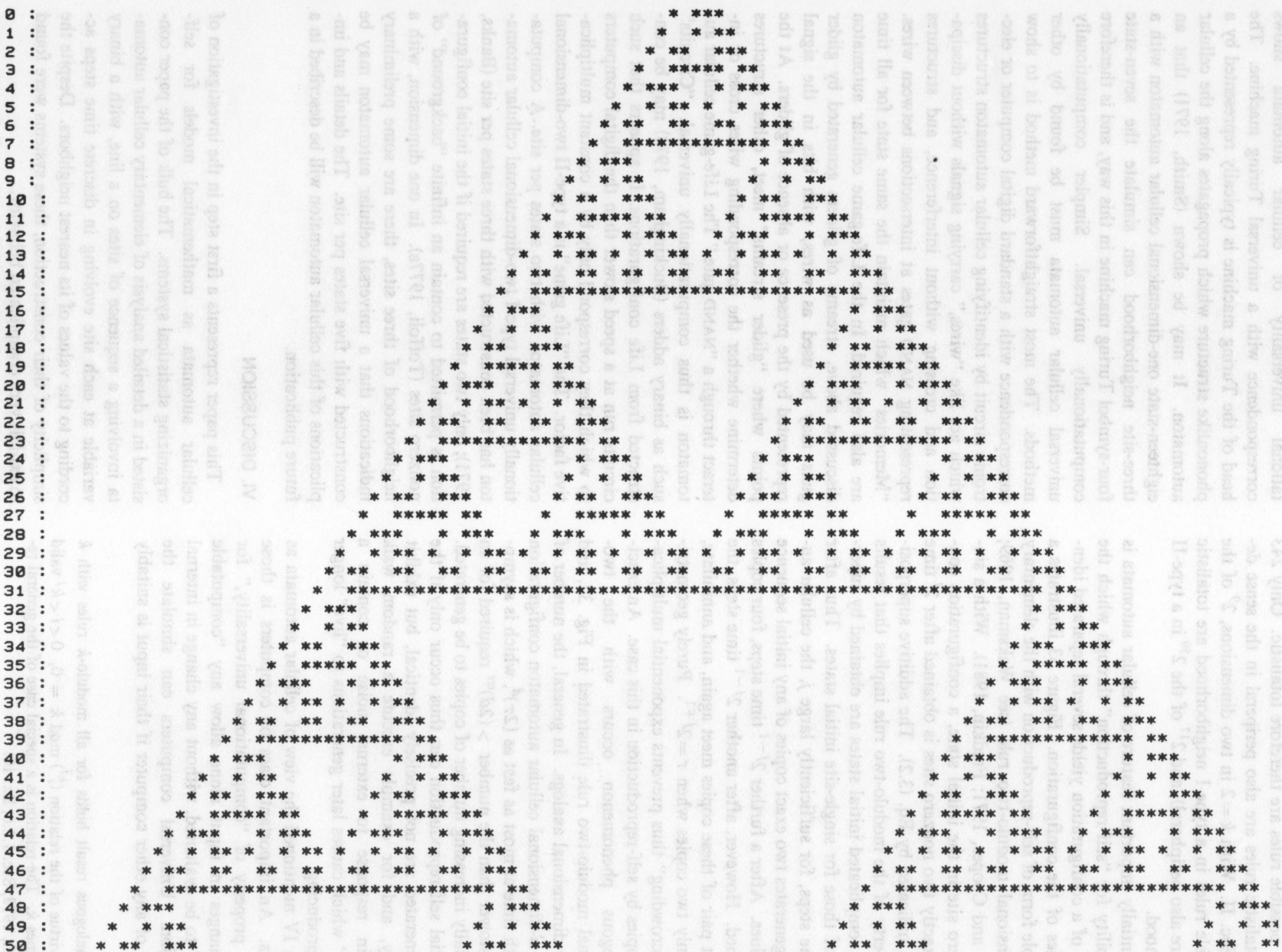
FIG. 33. Evolution of a simple pattern according to the modulo-two cellular automaton rule (number 90), exhibiting a simple self-reproduction phenomenon. The additive superposition property of the cellular automaton leads to the generation of two exact copies of the initial 1011 pattern at time steps 8, 16, 32, ... "Geometrical overcrowding" prevents exponential increase in the number of copies produced.

tion led eventually to simple, usually homogeneous, final states. The second class contained complex cellular automata capable of generating quite complicated structures even from simple initial states. Figure 3 showed the patterns of growth obtained with the very simplest initial state in which only one site had a nonzero value. The complex rules were found to yield self-similar fractal patterns. For all but one of the rules, the patterns exhibited the same fractal dimension $\log_2 3 \simeq 1.59$ (the remaining rule gave a fractal dimension $\log_2 2\varphi \simeq 1.69$). With more complicated initial states, the patterns obtained after evolution for many time steps remained self-similar—at least on scales larger than the region of nonzero initial sites. The generation of self-similar patterns was thus found to be a generic feature of complex cellular automata evolving from simple initial states. This result may provide some explanation for the widespread occurrence of self-similarity in natural systems.

Section III discussed the evolution of cellular automata from general initial states, in which a finite fraction of the infinite number of initial sites carried value one. Regardless of the initial density of nonzero sites, definite densities were found in the large time limit. Markovian master equation approximations to the density development were found inadequate because of the importance of "feedback" in the cellular automaton evolution. Even with disordered or random initial states, in which the values of different sites are statistically uncorrelated, the evolution of complex cellular automata was found to lead to the formation of definite structures, as suggested by Figs. 8 and 15. One characteristic of this self-organization was the generation of long sequences of correlated sites. The spectrum of these sequences was found to reach an equilibrium form after only a few time steps, extending to arbitrarily large scales, but with an exponential damping. The exponents were again found to be universal for all initial states and almost all complex cellular automata (with the exception of two special additive cellular automata).

Any initial cellular automaton state was found to lead at large times to configurations with the same statistical structures. However, in complex cellular automata, the trajectories of almost all specific nearby initial configurations (differing by changes in the values at a few sites) were found to diverge exponentially with time in the phase space of possible configurations. After a few time steps, the mapping from initial to final configurations becomes apparently random (although there are quantitative deviations from a uniform random mapping). Cellular automaton rules may map several initial configurations into the same final configuration, and thus lead to microscopically irreversible time evolution in which trajectories of different states may merge. In the limit of an infinite number of sites, a negligible fraction of all the possible cellular automaton configurations are reached by evolution from any of the possible initial states after a few time steps. Starting even from an ensemble in which each possible configuration appears with equal probability, the cellular automaton evolution concentrates the probabilities for particular configurations, thereby reducing entropy. This phenomenon allows for the possibility of self-organization by enhancing the probabilities of organized configurations and suppressing disorganized configurations.

Many of the qualitative features found for elementary cellular automata appear to survive in more complicated cellular automata (considered briefly in Sec. V), although several novel phenomena may appear. For example, in one-dimensional cellular automata with three or more possible values at each site, protective membranes may be generated which shield finite regions from the effects of external noise, and allow very regular patterns to grow from small seeds.

Cellular automata may be viewed as computers, with initial configurations considered as input programs and data processed by cellular automaton time evolution. Sufficiently complicated cellular automata are known to be universal computers, capable of computing any computable function given appropriate input. Such cellular automata may be considered as capable of the most complicated behavior conceivable and are presumably capable of simulating any physical system given a suitable input encoding and a sufficiently long running time. In addition, they may be used to simulate the evolution of any other cellular automaton. If the necessary encoding is sufficiently simple, the statistical properties of the simulated cellular automaton should follow those of the universal cellular automaton. Although not capable of universal simulation, simpler cellular automata may often simulate each other. This capability may well form a basis for the universality found in the statistical properties of various cellular automata.

Cellular automata have been developed in this paper as general mathematical models. One may anticipate their application as simple models for a wide variety of natural processes. Their nontrivial features are typically evident only when some form of growth inhibition is present. Examples are found in aggregation processes in which aggregation at a particular point prevents further aggregation at the same point on the next time step.

ACKNOWLEDGMENTS

REFERENCES

Abelson, H. and A. A. diSessa, 1981, *Turtle Geometry: The Computer as a Medium for Exploring Mathematics* (MIT Press, Cambridge).

Aggarwal, S., 1973, "Local and global Garden of Eden theorems," University of Michigan technical report No. 147.

Aladyev, V., 1974, "Survey of research in the theory of homogeneous structures and their applications," Math. Biosci. **22**, 121.

Aladyev, V., 1976, "The Behavioural Properties of Homogeneous Structures," Math. Biosci. **29**, 99.

Alekseev, V. M., and M. V. Yakobson, 1981, "Symbolic dynamics and hyperbolic dynamic systems," Phys. Rep. **75**, 287.

Amoroso, S. and G. Cooper, 1971, "Tessellation structures of reproduction of arbitrary patterns," J. Comput Syst. Sci. **5**, 455.

Apostol, T. M., 1976, *Introduction to Analytic Number Theory* (Springer, Berlin).

ApSimon, H. G., 1970a, "Periodic forests whose largest clearings are of size 3," Philos. Trans. R. Soc. London, Ser. A **266**, 113.

ApSimon, H. G., 1970b, "Periodic forests whose largest clearings are of size $n \geq 4$," Proc. R. Soc. London, Ser. A **319**, 399.

Arbib, M. A., 1969, *Theories of Abstract Automata* (Prentice-Hall, Englewood Cliffs).

Atrubin, A. J., 1965, "A one-dimensional real-time iterative multiplier," IEEE Trans. Comput. **EC-14**, 394.

Baer, R. M., and H. M. Martinez, 1974, "Automata and biology," Ann. Rev. Biophys. **3**, 255.

Banks, E. R., 1971, "Information processing and transmission in cellular automata," MIT Project MAC report No. TR-81.

Barricelli, N. A., 1972, "Numerical testing of evolution theories," J. Statist. Comput. Simul. **1**, 97.

Berlekamp, E. R., 1968, *Algebraic Coding Theory* (McGraw-Hill, New York).

Berlekamp, E. R., J. H. Conway, and R. K. Guy, 1982, *Winning Ways for Your Mathematical Plays* (Academic, New York), Vol. 2, Chap. 25.

Buckingham, D. J., 1978, "Some facts of life," Byte **3**, 54.

Burks, A. W., 1970, *Essays on Cellular Automata* (University of Illinois, Urbana).

Burks, A. W., 1973, "Cellular Automata and Natural Systems," Proceedings of the 5th Congress of the Deutsche Gessellschaft für Kybernetik, Nuremberg.

Butler, J. T., and S. C. Ntafos, 1977, "The vector string descriptor as a tool in the analysis of cellular automata systems," Math. Biosci. **35**, 55.

Codd, E. F., 1968, *Cellular Automata* (Academic, New York).

Cole, S. N., 1969, "Real-time computation by n-dimensional iterative arrays of finite-state machines," IEEE Trans. Comput. **C-18**, 349.

Conway, J. H., 1970, unpublished.

Deutsch, E. S., 1972, "Thinning algorithms on rectangular, hexagonal and triangular arrays," Commun. ACM **15**, 827.

Farmer, J. D., 1982a, "Dimension, fractal measures, and chaotic dynamics," in *Evolution of Order and Chaos in Physics, Chemistry and Biology*, edited by H. Haken (Springer, Berlin).

Farmer, J. D., 1982b, "Information dimension and the probabilistic structure of chaos," Z. Naturforsch. **37a**, 1304.

Fine, N. J., 1947, "Binomial coefficients modulo a prime," Am. Math. Mon. **54**, 589.

Fischer, P. C., 1965, "Generation of primes by a one-dimensional real-time iterative array," J. ACM **12**, 388.

Flanigan, L. K., 1965, "An experimental study of electrical conduction in the mammalian atrioventricular node," Ph.D. thesis (University of Michigan).

Fredkin, E., 1981, unpublished, and PERQ computer demonstration (Three Rivers Computer Corp.).

Gach, P., G. L. Kurdyumov, and L. A. Levin, 1978, "One-dimensional uniform arrays that wash out finite islands," Probl. Peredachi. Info., **14**, 92.

Gardner, M., 1971, "Mathematical Games," Sci. Amer. **224**, February, 112; March, 106; April, 114.

Gardner, M., 1972, "Mathematical Games," Sci. Amer. **226**, January, 104.

Geffen, Y., A. Aharony, B. B. Mandelbrot, and S. Kirkpatrick, "Solvable fractal family, and its possible relation to the backbone at percolation," Phys. Rev. Lett. **47**, 1771.

Gerola, H. and P. Seiden, 1978, "Stochastic star formation and spiral structure of galaxies," Astrophys. J. **223**, 129.

Glaisher, J. W. L., 1899, "On the residue of a binomial-theorem coefficient with respect to a prime modulus," Q. J. Math. **30**, 150.

Golomb, S. W., 1967, *Shift Register Sequences* (Holden-Day, San Francisco).

Grassberger, P., 1982, "A new mechanism for deterministic diffusion," Wuppertal preprint WU B 82-18.

Greenberg, J. M., B. D. Hassard, and S. P. Hastings, 1978, "Pattern formation and periodic structures in systems modelled by reaction-diffusion equations," Bull. Am. Math. Soc. **84**, 1296.

Griffeath, D., 1970, *Additive and Cancellative Interacting Particle Systems* (Springer, Berlin).

Haken, H., 1975, "Cooperative phenomena in systems far from thermal equilibrium and in nonphysical systems," Rev. Mod. Phys. **47**, 67.

Haken, H., 1978, *Synergetics*, 2nd ed. (Springer, Berlin).

Haken, H., 1979, *Pattern Formation by Dynamic Systems and Pattern Recognition* (Springer, Berlin).

Haken, H., 1981, *Chaos and Order in Nature* (Springer, Berlin).

Hardouin-Duparc, J., 1974, "Paradis terrestre dans l'automate cellulaire de conway," R.A.I.R.O. 8 R-3, 63.

Hardy, G. H. and E. M. Wright, 1979, *An Introduction to the Theory of Numbers*, 5th ed. (Oxford University Press, Oxford).

Harris, B., 1960, "Probability distributions related to random mappings," Ann. Math. Stat. **31**, 1045.

Harvey, J. A., E. W. Kolb, and S. Wolfram, 1982, unpublished.

Herman, G. T., 1969, "Computing ability of a developmental model for filamentous organisms," J. Theor. Biol. **25**, 421.

Honsberger, R., 1976, "Three surprises from combinatorics and number theory," in *Mathematical Gems II*, Dolciani Math. Expositions (Mathematical Association of America, Oberlin), p.1.

Hoogland, A., *et al.*, 1982, "A special-purpose processor for the Monte Carlo simulation of Ising spin systems," Delft preprint.

Hopcroft, J. E., and J. D. Ullman, 1979, *Introduction to Automata Theory, Languages and Computation* (Addison-Wesley, Reading).

Kauffman, S. A., 1969, "Metabolic stability and epigenesis in randomly constructed genetic nets," J. Theor. Biol. **22**, 437.

Kimball, S. H., *et al.*, 1958, "Odd binomial coefficients," Am. Math. Mon. **65**, 368.

Kitagawa, T., 1974, "Cell space approaches in biomathematics," Math. Biosci. **19**, 27.

Knuth, D. E., 1973, *Fundamental Algorithms* (Addison-Wesley, Reading).

Knuth, D. E., 1981, *Seminumerical Algorithms*, 2nd ed. (Addison-Wesley, Reading).

Kosaraju, S. R., 1974, "On some open problems in the theory of cellular automata," IEEE Trans. Comput. C-23, 561.

Landauer, R., 1979, "The role of fluctuations in multistable systems and in the transition to multistability," Ann. N.Y. Acad. Sci. 316, 433.

Langer, J. S., 1980, "Instabilities and pattern formation in crystal growth," Rev. Mod. Phys. 52, 1.

Levy, Y. E., 1982, "Some remarks about computer studies of dynamical systems," Phys. Lett. A 88, 1.

Lifshitz, E. M., and L. P. Pitaevskii, 1981, *Physical Kinetics* (Pergamon, New York).

Lindenmayer, A., 1968, "Mathematical models for cellular interactions in development," J. Theoret. Biol. 18, 280.

MacWilliams, F. J., and N. J. A. Sloane, *Theory of Error-Correcting Codes* (North-Holland, Amsterdam).

Mandelbrot, B., 1977, *Fractals: Form, Chance and Dimension* (Freeman, San Francisco).

Mandelbrot, B., 1982, *The Fractal Geometry of Nature* (Freeman, San Francisco).

Manna, Z., 1974, *Mathematical Theory of Computation* (McGraw-Hill, New York).

Manning, F. B., 1977, "An approach to highly integrated, computer-maintained cellular arrays," IEEE Trans. Comput. C-26, 536.

Margolus, N., 1982, private communication.

Martin, O., A. Odlyzko, and S. Wolfram, 1983, "Algebraic properties of cellular automata," Bell Laboratories report (January, 1983).

McIlroy, M. D., 1974, "The numbers of 1's in binary integers: bounds and extremal properties," SIAM J. Comput. 3, 255.

Miller, J. C. P., 1970, "Periodic forests of stunted trees," Philos. Trans. R. Soc. London, Ser. A 266, 63.

Miller, J. C. P., 1980, "Periodic forests of stunted trees," Philos. Trans. R. Soc. London Ser. A 293, 48.

Minsky, M. L., 1967, *Computation: Finite and Infinite Machines* (Prentice-Hall, Englewood Cliffs).

Moore, E. F., 1962, "Machine Models of Self-Reproduction," Proceedings of a Symposium on Applied Mathematics 14, 17, reprinted in *Essays on Cellular Automata*, edited by A. W. Burks (University of Illinois, Urbana, 1970), p. 187.

Nicolis, G. and Prigogine, I., 1977, *Self-Organization in Nonequilibrium Systems* (Wiley, New York).

Nicolis, G., G. Dewel, and J. W. Turner, editors, 1981, *Order and Fluctuations in Equilibrium and Nonequilibrium Statistical Mechanics*, Proceedings of the XVIIth International Solvay Conference on Physics (Wiley, New York).

Nishio, H., 1981, "Real time sorting of binary numbers by 1-dimensional cellular automata," Kyoto University report.

Ott, E., 1981, "Strange attractors and chaotic motions of dynamical systems," Rev. Mod. Phys. 53, 655.

Pearson, R., J. Richardson, and D. Toussaint, 1981, "A special purpose machine for Monte-Carlo simulation," Santa Barbara preprint NSF-ITP-81-139.

Peterson, W. W., and E. J. Weldon, *Error-Correcting Codes*, 2nd ed. (MIT Press, Cambridge).

Preston, K., M. J. B. Duff, S. Levialdi, Ph. E. Norgren, and J.-I. Toriwaki, 1979, "Basics of Cellular Logic with Some Applications in Medical Image Processing," Proc. IEEE 67, 826.

Prigogine, I., 1980, *From Being to Becoming* (Freeman, San Francisco).

Renyi, A., 1970, *Probability Theory* (North-Holland, Amsterdam).

Roberts, J. B., 1957, "On Binomial Coefficient Residues," Can. J. Math. 9, 363.

Rosen, R., 1981, "Pattern Generation in Networks," Prog. Theor. Biol. 6, 161.

Rosenfeld, A., 1979, *Picture Languages* (Academic, New York).

Schewe, P. F., editor, 1981, "Galaxies, the Game of Life, and Percolation," in *Physics News*, Amer. Inst. Phys. Pub. R-302, 61.

Schulman, L. S., and P. E. Seiden, 1978, "Statistical mechanics of a dynamical system based on Conway's game of life," J. Stat. Phys. 19, 293.

Shannon, C. E., 1951, "Prediction and entropy of printed English," Bell Syst. Tech. J., 30, 50.

Sierpinski, W., 1916, "Sur une courbe dont tout point est un point de ramification," Pr. Mat.-Fiz. 27, 77; *Oeuvres Choisis*, (Państowe Wydawnictwo Naukowe, Warsaw) Vol. II, p. 99.

Smith, A. R., 1971, "Simple computation-universal cellular spaces," J. ACM 18, 339.

Sternberg, S. R., 1980, "Language and architecture for parallel image processing," in *Pattern Recognition in Practice*, edited by E. S. Gelesma and L. N. Kanal (North-Holland, Amsterdam), p. 35.

Stevens, P. S., 1974, *Patterns in Nature* (Little, Brown, Boston).

Stolarsky, K. B., 1977, "Power and exponential sums of digital sums related to binomial coefficient parity," SIAM J. Appl. Math. 32, 717.

Sutton, C., 1981, "Forests and numbers and thinking backwards," New Sci. 90, 209.

Thom, R., 1975, *Structural Stability and Morphogenesis* (Benjamin, New York).

Thompson, D'A. W., 1961, *On Growth and Form*, abridged ed. edited by J. T. Bonner (Cambridge University, Cambridge, England).

Toffoli, T., 1977a, "Computation and construction universality of reversible cellular automata," J. Comput. Sys. Sci. 15, 213.

Toffoli, T., 1977b, "Cellular automata mechanics," Ph.D. thesis, Logic of Computers Group, University of Michigan.

Toffoli, T., 1980, "Reversible computing," MIT report MIT/LCS/TM-151.

Toffoli, T., 1983, "Squareland: a hardware cellular automaton simulator," MIT LCS preprint, in preparation.

Turing, A. M., 1936, "On computable numbers, with an application to the Entscheidungsproblem," Proc. London Math. Soc. Ser. 2, 42, 230; 43, 544E, reprinted in *The Undecidable*, edited by M. David (1965; Hewlett, New York), p. 115.

Turing, A. M., 1952, "The chemical basis of morphogenesis," Philos. Trans. R. Soc. London, Ser. B 237, 37.

Ulam, S., 1974, "Some ideas and prospects in biomathematics," Ann. Rev. Bio., 255.

von Neumann, J., 1963, "The general and logical theory of automata," in J. von Neumann, *Collected Works*, edited by A. H. Taub, 5, 288.

von Neumann, J., 1966, *Theory of Self-Reproducing Automata*, edited by A. W. Burks (University of Illinois, Urbana).

Wainwright, R. T., 1971—73, Lifeline, 1—11.

Wainwright, R. T., 1974, "Life is Universal!," Proceedings of the Winter Simulation Conference, Washington, D.C., ACM, p. 448.

Waksman, A., 1969, "A model of replication," J. ACM 16, 178.

Willson, S., 1982, "Cellular automata can generate fractals," Iowa State University, Department of Mathematics, preprint.

Witten, T. A. and L. M. Sander, 1981, "Diffusion-limited aggregation, a kinetic critical phenomenon," Phys. Rev. Lett. **47**, 1400.

Wolfram, S., *et al.*, 1981, "SMP Handbook," Caltech.

Wolfram, S., 1982a, "Cellular automata as simple self-organizing systems," Caltech preprint CALT-68-938 (submitted to Nature).

Wolfram, S., 1982b, "Geometry of binomial coefficients," to be published in Am. Math. Monthly.